Week 12-Lab 2: Worksheet 17: Section 16.1

I said: " This worksheet and what comes after this are very important for your future physics and electromagnetism courses."

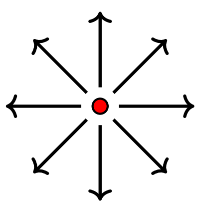
Divergence, Gradient, and Curl in cylindrical coordinates:

Gradient: $\nabla f \;=\frac{\partial f}{\partial x}\vec{\mathbf{i}}+\frac{\partial f}{\partial y}\vec{\mathbf{j}}+\frac{\partial f}{\partial z}\vec{\mathbf{k}}$

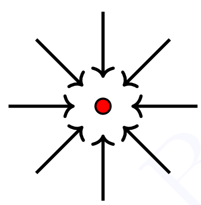
Divergence: $\nabla\cdot\vec{\mathbf{F}} \;=\frac{\partial\left(F_x\right)}{\partial x}+\frac{\partial F_y}{\partial y}+\frac{\partial F_z}{\partial z}$ **Curl:** $\nabla\times\vec{\mathbf{F}} \;=\begin{vmatrix}\vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ F_x & F_y & F_z\end{vmatrix}$

Divergence of a vector field:

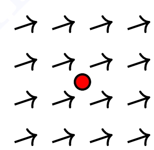
The divergence of a vector field $\vec{\mathbf{F}}$ at a point P measures how much $\vec{\mathbf{F}}$ disperses "stuff" near P. Div($\vec{\mathbf{F}}$) is a <u>scalar function</u>.



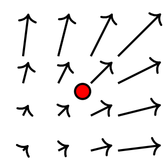
$\mathrm{div}(\vec{\mathbf{F}})>0$
disperses stuff (source)



$\mathrm{div}(\vec{\mathbf{F}})<0$
attracts stuff (sink)



$\mathrm{div}(\vec{\mathbf{F}})=0$
"incompressible"

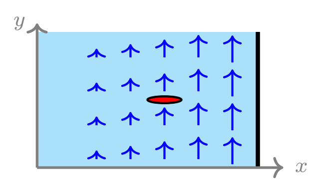


$\mathrm{div}(\vec{\mathbf{F}})>0$

Curl of a vector field:

The **curl** of a <u>vector field</u> $\vec{\mathbf{F}}$ measures how $\vec{\mathbf{F}}$ causes objects to rotate.

Example: The current in a river is stronger near the banks than in the middle. A boat is anchored near the right bank. What happens to the boat? **It rotates counterclockwise.**



Curl($\vec{\mathbf{F}}$) >0 is a <u>vector field</u> sticking out of the page toward us.

Physical properties of Div., Grad. and Curl:

Gradient of a vector field: Let $f(x,y,z)$ be a scalar-valued function. Its gradient is a vector field:

$$\vec{\mathbf{F}}=\nabla f=\left\langle\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z}\right\rangle$$

- The function f is called a **(scalar) potential function** for $\vec{\mathbf{F}}$.
- A vector field is called **conservative** if it has a potential function,

A few properties:

$\nabla \times (\nabla f) = \vec{0}$ or curl of gradient is zero. $\nabla \cdot (\nabla \times \vec{\mathbf{F}}) = 0$ or divergence of the curl $= 0$.

Conservative Vector Fields:

Let $f(x, y, z)$ be a scalar-valued function. Its gradient is a vector field:

$$\vec{\mathbf{F}} = \nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

The function f is called a **(scalar) potential function** for $\vec{\mathbf{F}}$.

A vector field is called **conservative** if it has a potential function,

Conservative fields occur naturally in physics, as force fields in which energy is conserved.

Theorem:

If $\vec{\mathbf{F}}$ is conservative on an open connected domain $\mathcal{R}$, then any two potential functions of $\vec{\mathbf{F}}$ differ by a constant.

Theorem:

If $\vec{\mathbf{F}}$ is a conservative vector field in $\mathbb{R}^2$ or $\mathbb{R}^3$, then $\text{Curl}(\vec{\mathbf{F}}) = \vec{0}$.

Finding Scalar Potentials The process for finding scalar potential functions is essentially antidifferentiation, but with a twist.

For $\vec{\mathbf{F}}(x, y) = \langle F_1(x, y),\ F_2(x, y)\rangle$:

1. Find the indefinite integrals $\int F_1(x, y)\, dx$ and $\int F_2(x, y)\, dy$.
   - **The constants of integration are $c_1(y)$ and $c_2(x)$ respectively** (instead of the usual "$+C$"), because if $\frac{\partial}{\partial x}(f(x, y)) = F_1$ then $\frac{\partial}{\partial x}(f(x, y) + c_1(y)) = F_1$ as well.
2. "Match up the pieces" to determine $f(x, y)$.

For $\vec{\mathbf{F}}(x, y, z) = \langle F_1(x, y, z),\ F_2(x, y, z),\ F_3(x, y, z)\rangle$:

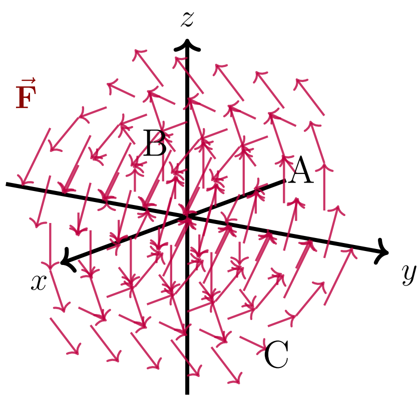
1. Find the indefinite integrals $\int F_1\, dx$, $\int F_2\, dy$, and $\int F_3\, dz$.
   Constants of integration: $c_1(y, z)$, $c_2(x, z)$, $c_3(x, y)$.
2. "Match up the pieces" to determine $f(x, y, z)$.

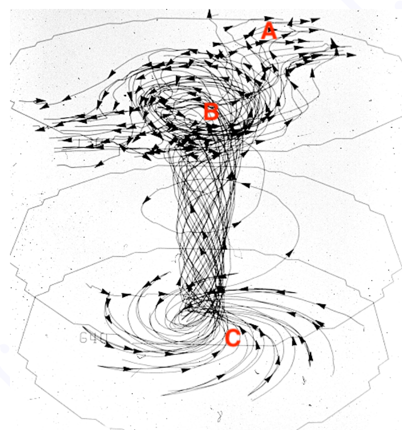
Group Work Portion of Worksheet

Names: ____

Work in groups to do this portion of the worksheet. Make sure to take parts in solving the problems. Your participation score is a combination of being prepared, willing to explore the problem, working in groups and contributing toward the solution.

1. Use the right hand rule and draw a vector in the direction of curl at points A, B and C for each vector field.



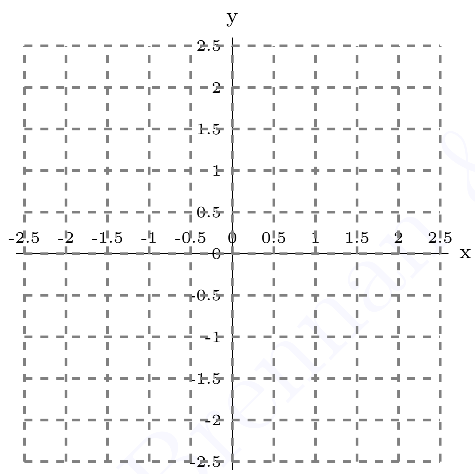


Vector Field $\vec{\mathbf{T}}$: Airflow of a Tornado

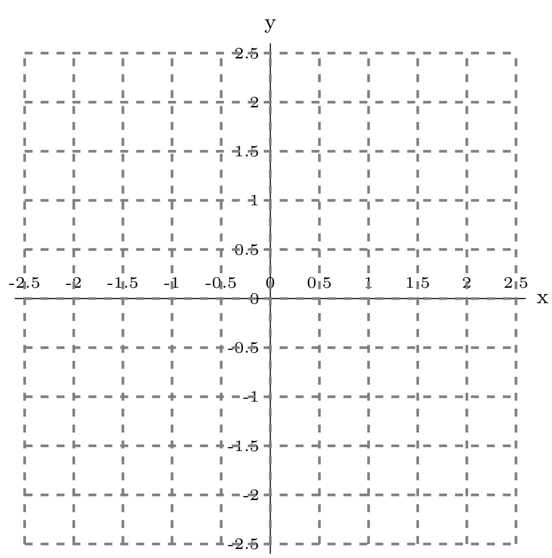
2. For each of following vector fields,

   (i) sketch the vector field,
   (ii) compute **curl** of the vector field
   (iii) compute **divergence** of the vector field
   (iv) explain why the sign of computed curl matches what you expected on the sketch
   (v) explain if point $(1,1)$ is a source, a sink or incompressible for the vector field
   (vi) explain if the vector field has a potential and if it does, find a potential function.

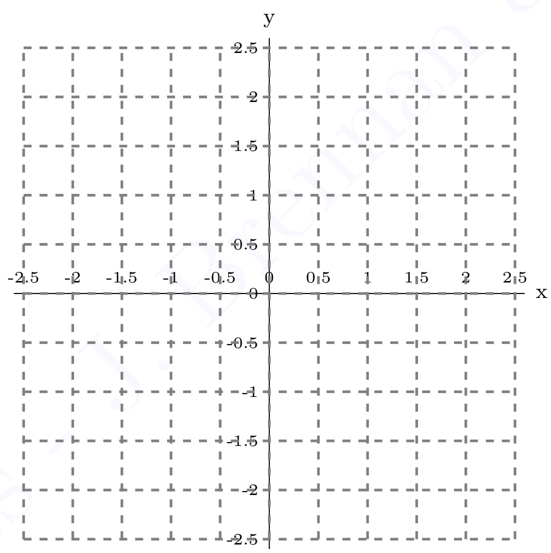
   (A) $\vec{\mathbf{A}}(x,y) = \langle -4y-3, y+5\rangle$.



   (B) $\vec{\mathbf{B}}(x,y) = \langle 2y-x, -1.5\rangle$.



(C) $\vec{\mathbf{C}} = \nabla f$ where $f(x, y) = xy$.



Video: https://youtu.be/Et1wqoSvaVE and https://mediahub.ku.edu/media/t/1_00lmckae

3. **Background Story:** Let's discuss a vector field with many applications in physics and electromagnetism.

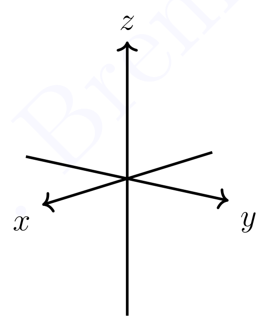
Question(s):

(A) Let be the scalar-valued function of three variables $r = \sqrt{x^2 + y^2 + z^2}$. Find ∇r.

(B) Find the domain of ∇r.

(C) Evaluate $\|\nabla r\|$.

(D) Remember for a point $P = (x, y, z)$, the position vector is $\vec{\mathbf{r}} = \langle x, y, z \rangle$. Denote $\vec{e}_r = \dfrac{\vec{\mathbf{r}}}{\|\vec{\mathbf{r}}\|}$. Graph the vector field $\vec{e}_r$ on it's domain. (Draw two vectors in each octant.) [1]



(E) Find a scalar potential function for $\vec{e}_r$? (Hint: You know this from previous parts.)

(F) Find $\text{curl}(\vec{e}_r)$.

(G) Find $\text{div}(\vec{e}_r)$.

[1] Vector fields $\vec{\mathbb{E}} = k\vec{e}_r$, where k is a constant, have many applications in physics and will be denoted differently in different area of science. This exercise helps you know their scalar potential function and properties.

4. **Background Story:** Come up with the steps on your own. The Individual portion of worksheet contains similar problem(s).

   Question(s): Let $\vec{F}(x,y,z) = \langle z\cos(y), -xz\sin(y) + e^y, x\cos(y)\rangle$. Is $\vec{F}$ conservative? If it is conservative, then find a potential function. If it is not conservative, then calculate $\operatorname{curl}\left(\vec{F}\right)$.

Video: https://youtu.be/Eq9JezYXZRQ

GroupWork Rubrics:

Preparedness: ——/0.5, Contribution: ——/0.5, Correct Answers: ——/0.5

Individual Portion of Worksheet

Name: ________________

Upload this section individually on canvas or turn it in to your instructor on the 2^{nd} lab day of the week. You can ask questions in class and work in groups but you turn in the individual work. Start before the class so you can ask questions during the class. If you didn't complete the work in class, make sure to work on it outside the class and complete it. Show all your work; your score depends on the work you have shown.

GroupWork Rubrics day 2:

Preparedness: ——/0.5, Contribution: ——/0.5, Correct Answers: ——/0.5

5. Let $\vec{F}(x, y, z) = \langle 2xy, x^2 - 4z, -4y + 9 \rangle$.

(A) *(0.5 points)* Calculate $\operatorname{curl}\left(\vec{F}\right)$.

(B) *(0.5 points)* Is $\vec{F}$ conservative?

(C) *(2.5 points)* If it is conservative, then find a potential function. If it is not conservative, explain why.

6. **Background Story:** You have seen this function earlier in this worksheet.

Question(s):

Let $\vec{e}_r = \left\langle \dfrac{x}{\sqrt{x^2+y^2+z^2}}, \dfrac{y}{\sqrt{x^2+y^2+z^2}}, \dfrac{z}{\sqrt{x^2+y^2+z^2}} \right\rangle$.

(A) *(1.5 points)* Find $\displaystyle\int \frac{x}{\sqrt{x^2+y^2+z^2}}dx$, $\displaystyle\int \frac{y}{\sqrt{x^2+y^2+z^2}}dy$ and $\displaystyle\int \frac{z}{\sqrt{x^2+y^2+z^2}}dz$.

(B) *(0.5 points)* Is the domain of $\vec{e}_r$ an open and connected domain?

(C) *(1.5 points)* Find the general scalar potential function for $\vec{e}_r$.