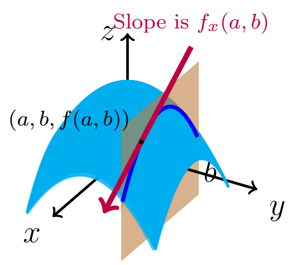
Week 5-Lab 1: Worksheet 6: Section 14.3, 14.4 and 14.5

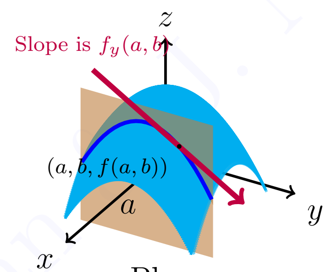
I said: "Do you know we have a help room in 651 Snow Hall where your instructors can help you? Our hours are M-F Noon-5PM." I also said: " Don't get too much help on assignments! Get as much as possible done. Then ask the instructors to lead you to the next step if needed. "

14.3: Geometry of Partial Derivatives:

The planes $x = a$ and $y = b$ intersect the surface $z = f(x, y)$ in curves $z = f(a, y)$ and $z = f(x, b)$ (respectively). The partial derivatives are the slopes of the tangent lines to the two curves.



Plane $y = b$



Plane $x = a$

https://youtu.be/b52bcTlWtFs

- The tangent line to the graph of $z = f(x, b)$ contains the point $(a, b, f(a, b))$ and has direction vector $\langle 1, 0, f_x(a, b)\rangle$.
- The tangent line to the graph of $z = f(a, y)$ contains the point $(a, b, f(a, b))$ and has direction vector $\langle 0, 1, f_y(a, b)\rangle$.

Clairaut's Theorem

If $f_{xy}(x, y)$ and $f_{yx}(x, y)$ are continuous, then $f_{xy} = f_{yx}$.

14.4: Algebraic Definition of Differentiability:

Suppose that $(x, y) = (a, b)$ is in the domain of a function $z = f(x, y)$. We know that the tangent plane, <u>if it exists,</u> has the equation

$$L_{(a,b)}(x, y) = z = f_x(a, b)(x - a) + f_y(a, b)(y - b) + f(a, b).$$

Total Differentials

$$dz = f_x(a, b)\Delta x + f_y(a, b)\Delta y$$

How the Differentials and Tangent the Plane are Related

$$\begin{aligned} L_{(a,b)}(x, y) = z \quad &= f_x(a, b)\underbrace{(x - a)}_{\Delta x} + f_y(a, b)\underbrace{(y - b)}_{\Delta y} + f(a, b) \\ \underbrace{z - f(a, b)}_{dz} \quad &= f_x(a, b)\underbrace{(x - a)}_{\Delta x} + f_y(a, b)\underbrace{(y - b)}_{\Delta y} \end{aligned}$$

14.5: Gradients and Directional Derivatives in 2 Variables

https://youtu.be/AJcMBgbkkws

Let $f(x, y)$ be a scalar function of 2 variables. The **gradient vector** of differentiable function f at a point (a, b) in the domain of f is $\nabla f(a, b) = \langle f_x(a, b),\ f_y(a, b)\rangle$.

If $\vec{\mathbf{u}} = \langle p, q\rangle$ is a unit vector, then the **directional derivative** of f at (a, b) in the direction of $\vec{\mathbf{u}}$ is

$$D_{\vec{\mathbf{u}}} f(a, b) = \nabla f(a, b) \cdot \vec{\mathbf{u}}$$

Gradients and Directional Derivatives in 3 Variables

Let $f(x, y, z)$ be a function of 3 variables. The **gradient vector** of a scalar differentiable function f at a point (a, b, c) in the domain of f is

$$\nabla f(a, b, c) = \langle f_x(a, b, c),\ f_y(a, b, c),\ f_z(a, b, c)\rangle \ .$$

If $\vec{\mathbf{u}} = \langle p, q, r\rangle$ is a unit vector, then the **directional derivative** of f at (a, b, c) in the direction of $\vec{\mathbf{u}}$ is

$$D_{\vec{\mathbf{u}}} f(a, b, c) \cdot \nabla f(a, b, c) \cdot \vec{\mathbf{u}}$$

Directions with Extreme Rates of Change

The gradient ∇f points in the direction that f is increasing fastest.

– The **largest** (**smallest**) directional derivative is in the direction $\nabla f(a, b)$ (or $-\nabla f(a, b)$) and equal to $\|\nabla f(a, b)\|$ (or $-\|\nabla f(a, b)\|$).

Tangent Planes and Normal Lines

We can find the tangent plane to any surface S defined by an equation in x, y, z — we do not need z to be a function of x and y.

Express the equation in the form $F(x, y, z) = \text{constant}$

$\rightarrow$ The point: Now S is a level surface of F.

Next, compute $\nabla F(x, y, z)$.

Then, the equation of the tangent plane at any point (a, b, c) on S is

$$\nabla F(a, b, c) \cdot \langle x - a,\ y - b,\ z - c\rangle = 0$$

and the normal line has equation

$$\vec{\mathbf{r}}(t) = \langle a, b, c\rangle + t\nabla F(a, b, c)$$

or equivalently

$$x = a + tF_x(a, b, c), \quad y = b + tF_y(a, b, c), \quad z = c + tF_z(a, b, c).$$

Group Work Portion of Worksheet

Names: ____

Work in groups to do this portion of the worksheet. Make sure to take parts in solving the problems. Your participation score is a combination of being prepared, willing to explore the problem, working in groups and contributing toward the solution.

1. **Background Story:** A common error evaluating any derivative at a point is forgetting to plug in the numbers.

   Questions: A student was asked to find the equation of the tangent plane to the surface $z = x^3 - y^2$ at the point $(x, y) = (5, 2)$. The student answered:

   $$z = -2y(x-5) + 3x^2(y-2) + 121$$

   (A) At a glance, how do you know this is wrong?

   (B) What mistakes did the student make?

   (C) Answer the question correctly.

2. The temperature at a point (x, y) on a flat metal plate is given by

$$T(x, y) = \frac{75}{\sqrt{12 + x^2 + y^2}}$$

Where T is measured in $^\circ C$ and x, y in meters. Find the rate of change of temperature with respect to distance at the point $(2, 3)$ in the x-direction and the y-direction.

3. **Background Story:** Partial derivatives keep showing up in many areas. Here is an application example.

   Questions: At a distance of x feet from the beach, the price (\$) of a plot of land of area a square feet is $f(a, x)$.

   (i) What are the units of $f_a(a, x)$.

   (a) What does $f_a(1000, 470) = 7$ mean in practical terms?

   (b) What are the units of $f_x(a, x)$?

   (c) What does $f_x(1000, 470) = -4$ mean in practical terms?

   (d) Which is cheaper:

      (i) 1004 square feet that are 474 feet from the beach?

      (ii) 996 square feet that are 469 feet from the beach?

      Justify your answer.

4. **Background Story:**

(A) Compare the methods in Section 14.4 and 14.5 for finding the tangent plane to the graph of $z = f(x, y)$ at point $(a, b, f(a, b)$.

(Recap: In Section 14.4 we gave the formula $z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$ *and in Section 14.5, we introduced* $\nabla g(a, b, f(a, b))$ *as the normal vector to tangent plane at* $(a, b, f(a, b))$ *when* $g(x, y, z) = z - f(x, y)$*.)*

(B) Which method is preferred if the surface is not a graph of a function.

GroupWork Rubrics:

Preparedness: ——/0.5, Contribution: ——/0.5, Correct Answers: ——/0.5

Individual Portion of Worksheet

Name: ________________

Upload this section individually on canvas or turn it in to your instructor on the 2nd lab day of the week. You can ask questions in class and work in groups but you turn in the individual work. Start before the class so you can ask questions during the class. If you didn't complete the work in class, make sure to work on it outside the class and complete it. Show all your work; your score depends on the work you have shown.

GroupWork Rubrics day 2:

Preparedness: ——/0.5, Contribution: ——/0.5, Correct Answers: ——/0.5

5. *(1.5 points)* Compute the first-order partial derivatives of $w(x, y, z) = \dfrac{3y}{7x + 5z}$.

6. *(1 point)* Verify that u is a solution to the equation $u_{xx} + u_{yy} + u_{zz} = 0$.

$$u(x, y, z) = e^{4x+3y} \cos(5z)$$

[1]

[1] Video: https://youtu.be/TZrFe-aJf_U

7. (2.5 points) Approximate $\sqrt{(2.03)^2 + (0.8)^2 + (2.1)^2}$ using a linear approximation.[2]

8. **Background Story:** Now lets try to find the tangent plane to an implicit surface which is not graph of a function. Use gradient vector as the normal vector.

   Questions: *(2 points)* Find the tangent plane to the surface of $x^2 + z^2 = y^2 + 9$ at $(1, -1, 3)$. [3]

[2]Video: https://youtu.be/ck5GnrW1HkI
[3]Video: https://youtu.be/cOFFPMRez6s