Week 6-Lab 1: Worksheet 7: 14.5 and 14.6

She said: " I was solving the questions correctly for my group but every time, they preferred his answers over mine. He was giving the exact same answers." I said: " I am so sorry!" I asked my daughter: "What did you do when this happened to you?" She said: "I had an ally. He said that I did a better job answering the questions than he did. That reduced the pain." I am asking you: " Be an ally!"

Gradients and Directional Derivatives in 2 Variables

Let $f(x, y)$ be a scalar function of 2 variables. The **gradient vector** of differentiable function f at a point (a, b) in the domain of f is $\nabla f(a, b) = \langle f_x(a, b),\ f_y(a, b)\rangle$.

If $\vec{\mathbf{u}} = \langle p, q\rangle$ is a unit vector, then the **directional derivative** of f at (a, b) in the direction of $\vec{\mathbf{u}}$ is

$$D_{\vec{\mathbf{u}}} f(a, b) \cdot \nabla f(a, b) \cdot \vec{\mathbf{u}}$$

Gradients and Directional Derivatives in 3 Variables

Let $f(x, y, z)$ be a function of 3 variables. The **gradient vector** of a scalar differentiable function f at a point (a, b, c) in the domain of f is

$$\nabla f(a, b, c) = \langle f_x(a, b, c),\ f_y(a, b, c),\ f_z(a, b, c)\rangle \ .$$

If $\vec{\mathbf{u}} = \langle p, q, r\rangle$ is a unit vector, then the **directional derivative** of f at (a, b, c) in the direction of $\vec{\mathbf{u}}$ is

$$D_{\vec{\mathbf{u}}} f(a, b, c) \cdot \nabla f(a, b, c) \cdot \vec{\mathbf{u}}$$

Directions with Extreme Rates of Change

The gradient ∇f points in the direction that f is increasing fastest.

– The **largest** (**smallest**) directional derivative is in the direction $\nabla f(a, b)$ (or $-\nabla f(a, b)$) and equal to $\|\nabla f(a, b)\|$ (or $-\|\nabla f(a, b)\|$).

Tangent Planes and Normal Lines

We can find the tangent plane to any surface S defined by an equation in x, y, z — we do not need z to be a function of x and y.

Express the equation in the form $F(x, y, z) = \text{constant}$

$\rightarrow$ The point: Now S is a level surface of F.

Next, compute $\nabla F(x, y, z)$.

Then, the equation of the tangent plane at any point (a, b, c) on S is

$$\nabla F(a, b, c) \cdot \langle x - a,\ y - b,\ z - c\rangle = 0$$

and the normal line has equation

$$\vec{\mathbf{r}}(t) = \langle a, b, c\rangle + t\nabla F(a, b, c)$$

or equivalently

$$x = a + tF_x(a, b, c), \quad y = b + tF_y(a, b, c), \quad z = c + tF_z(a, b, c).$$

The Multivariable Chain Rule

Multivariable Chain Rule — First Case: Suppose that $z = f(x, y)$, $x = x(t)$, and $y = y(t)$ are differentiable functions. Then $z = g(t) = f(x(t), y(t))$ is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

In Lagrange notation, the formula is

$$g'(t) = f_x(x, y)x'(t) + f_y(x, y)y'(t)$$

Suppose $z = f(x, y)$, $x = g(s, t)$, and $y = h(s, t)$, with all functions differentiable. Then $z(s, t) = f\left(g(s, t), h(s, t)\right)$ is differentiable, and

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

For every composite function involving multiple variables, we can keep track of the Chain Rule using a tree.

Chain Rule Tree Diagram:

(1) Begin with the variable of the top of $\frac{d\square}{d\square}$ **or** $\frac{\partial\square}{\partial\square}$

(2) Relate the top variable to the next set of variable on the second line.

(3) Connect the second line variables to other variables they are immediately related to on the third line and so on until you arrive at the level that contains the bottom variable of Item (1) in this list.

(4) On each tree branch write $\frac{d\square}{d\square}$ **or** $\frac{\partial\square}{\partial\square}$ **relating the two variable that branch is connecting.**

(5) Multiply what you wrote on each branch that connects the two variable in Item (1). Then add all those numbers.

Group Work Portion of Worksheet

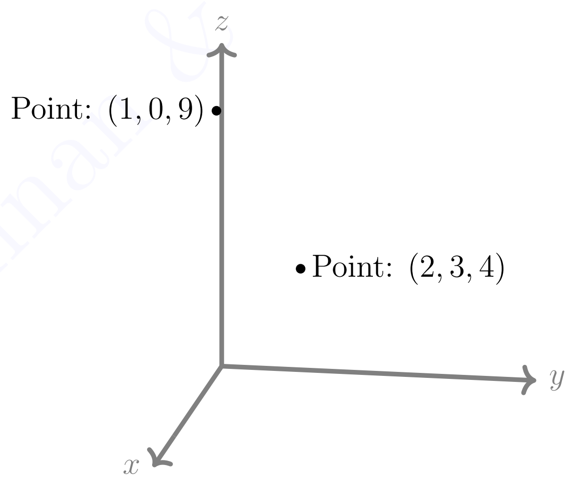
Names: ____

Work in groups to do this portion of the worksheet. Make sure to take parts in solving the problems. Your participation score is a combination of being prepared, willing to explore the problem, working in groups and contributing toward the solution.

1. **Background Story:** The gradient of a scalar function is perpendicular to level surfaces of that scalar function.

   Questions: Consider the differentiable scalar function $R(x, y, z) = \sqrt{x^2 + y^2 + z^2}$.

   (a) Describe and sketch the level surfaces for R at $(2, 3, 4)$ and $(1, 0, 9)$.

   (b) Compute and sketch the gradient vectors $\nabla R(2, 3, 4)$ and $\nabla R(1, 0, 9)$.

Video to similar Problem: https://youtu.be/NdeYuIi8TnQ

2. The temperature at a point (x, y, z) is given by

$$T(x, y, z) = \frac{1600e^{\frac{-5y^2}{10}}}{x^2 + 9z^2}$$

where T is measured in $^\circ C$ and x, y, z in meters.

(A) Find the gradient vector of T at $P(2, 0, 2)$.

(B) Find the rate of change of temperature at the point P in the direction toward the point $(1, 0, -1)$.

(C) In which direction does the temperature increase fastest at P?

(D) Find the fastest rate of temperature increase or decrease at P.

Video: https://youtu.be/qJ4M9Iisa3w

3. **Background Story:** The gradient of a level curve is orthogonal to the level curve containing it. In this question, explore the concepts.

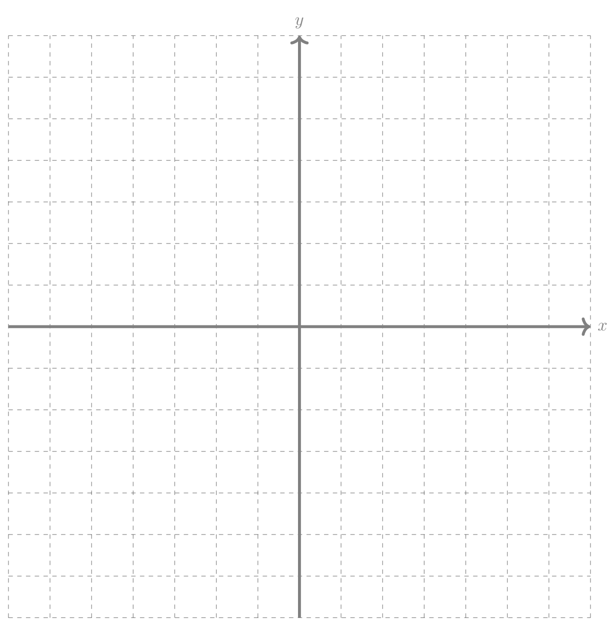
Questions: Consider $g(x, y) = x^2 + y - 4x$.

(A) Compute the gradient vector $\nabla g(3, 1)$.

(B) Is $(3, 1)$ on the level curve $g(x, y) = -2$?

(C) Use the $\nabla g(3, 1)$ to find the tangent line to the curve $g(x, y) = -2$ at $(3, 1)$.

(D) Sketch the level curve $g(x, y) = -2$ and the $\nabla g(3, 1)$ at point $(3, 1)$.



4. **Background Story:** Here is an application of chain rule. In designing a circuit with varying elements including an inductor, it is important to know how fast the current is changing. While the setting of the following problem is far from a design question for that, it can relay the information.

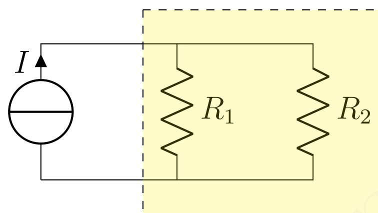
   Questions: The voltage V across a circuit is given by Ohm's law

$$V = IR$$

   where I is the current (in amps) flowing through the circuit and R is the resistance (in ohms). If we place two circuits, with resistance R_1 and R_2, in parallel, then their combined resistance is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

   Suppose the voltage drop on the resistance is 7 volts and increases at 10^{-2} volts/sec, R_1 is 5 ohms and decreases at 0.5 ohm/sec, and R_2 is 4 ohms and increases at 0.1 ohm/sec. Calculate the rate at which the current is changing.



$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

5. **Background Story:** The chain rule tree diagrams are really good tools for memorizing all chain rule formula. [1] Practice them here. Use your Section 14.6 lecture notes.

Questions:

(A) Draw a tree diagram for partials g_s and g_t when $g(s,t) = f(x(s,t), y(s,t))$. (Write a variable in each node and a partial derivative or a derivative on each branch. Highlight the branches used for g_s and g_t in different color if possible.)

(B) Discuss Question 3: Use

$$x(s,t) = 4s + 3t + 2st + 3 \quad \text{and} \quad y(s,t) = 7s + 3t^2 + 2.$$

and explain how to compute the highlighted numbers. Complete the individual part together.

(s,t)	x	x_s	x_t	y	y_s	y_t
$(0,0)$	3	4	3	2	7	0
$(3,2)$	33	a	b	35	7	12

[1] They take the memorization pain out of the memorization. 😄

GroupWork Rubrics:

Preparedness: ___/0.5, Contribution: ___/0.5, Correct Answers: ___/0.5

Individual Portion of Worksheet

Name: ________________

Upload this section individually on canvas or turn it in to your instructor on the 2nd lab day of the week. You can ask questions in class and work in groups but you turn in the individual work. Start before the class so you can ask questions during the class. If you didn't complete the work in class, make sure to work on it outside the class and complete it. Show all your work; your score depends on the work you have shown.

GroupWork Rubrics day 2:

Preparedness: ___/0.5, Contribution: ___/0.5, Correct Answers: ___/0.5

6. Suppose $f(x, y)$ is differentiable.

$$x(s,t) = 4s + 3t + 2st + 3 \quad \text{and} \quad y(s,t) = 7s + 3t^2 + 2.$$

We are given the value of the function $f(x, y)$ and it's partials at $(0, 0)$ and $(3, 2)$ in the table below. Let $g(s, t) = f(x(s, t), y(s, t))$.

	f	f_x	f_y
$(0,0)$	3	5	5
$(3,2)$	-4	3	2

(i) *(1 point)* Compute the missing values (a and b) in the table below using the $x(s, t)$ and $y(s, t)$.

(s,t)	x	x_s	x_t	y	y_s	y_t
$(0,0)$	3	4	3	2	7	0
$(3,2)$	33	a	b	35	7	12

(ii) *(1.25 points)* Find $g_s(0, 0)$.

(iii) *(1.25 points)* Find $g_t(0, 0)$.

Videos: https://youtu.be/aBKBd9VG1jA
https://youtu.be/Tn-RS0tsIfM