Week 10-Lab 1: Worksheet 14: Section 15.6

They said: "It feels like I am constantly bearing the entire work load for my group. They are not doing any work outside the class." I said: "Everyone is accountable for their work and they need to prepare. Divide the problems between your group ahead of time to reduce the workload for each individual member, then each member can take the lead in solving one problem; if that is not working neither, ask your instructor to switch groups."

Jacobians and the Change-Of-Variable Formula

The **Jacobian** of the transformation $\vec{\mathbf{G}}(u,v) = (x(u,v), y(u,v))$ is defined as

$$\left|\text{Jac}(\vec{\mathbf{G}})\right| = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \|\vec{\mathbf{G}}_u \times \vec{\mathbf{G}}_v\| = abs\left(\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}\right) = \left|\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right|$$

The absolute value of Jacobian is the area scaling factor for G. That is, the scaling factor is $\|\vec{\mathbf{G}}_u \times \vec{\mathbf{G}}_v\|$.

Double Integration with Change of Variables Let $\vec{\mathbf{G}}(u,v) = (x(u,v), y(u,v))$ be a transformation, and let $\vec{\mathbf{G}}(S) = R$. Then

$$\iint\limits_R f(x,y)\,dA_{xy} = \iint\limits_S f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dA_{uv}$$

Using this formula does not typically evaluate the integral immediately, but it enables you to convert it into an integral over a geometrically simpler region.

A Useful Fact About Jacobians

If $\vec{\mathbf{F}}$ is the inverse transformation of $\vec{\mathbf{G}}$, that is,

$$\vec{\mathbf{F}}(x,y) = (u,v) \quad \text{and} \quad \vec{\mathbf{G}}(u,v) = (x,y),$$

then

$$\text{Jac}(F) = \text{Jac}(G)^{-1}.$$

This fact is suggested by the notation:

$$\text{Jac}(\vec{\mathbf{G}}) = \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}} = \frac{1}{\text{Jac}(\vec{\mathbf{F}})}$$

Change of Variables for Triple Integrals

Let R be a region in $\mathbb{R}^3$ with coordinates x, y, z.
Let S be a region in $\mathbb{R}^3$ with coordinates u, v, w.
Let G be a transformation that maps S to R:

$$\vec{\mathbf{G}}(u,v,w) = \big(x(u,v,w),\, y(u,v,w),\, z(u,v,w)\big).$$

Then

$$\iiint\limits_R f(x,y,z)\,dV_{xyz} = \iiint\limits_S f\big(\vec{\mathbf{G}}(u,v,w)\big) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dV_{uvw}$$

where

$$\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = \left|\text{Jac}(\vec{\mathbf{G}})\right| = abs\left(\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}\right).$$

Group Work Portion of the Worksheet

Names: ____

Work in groups to do this portion of the worksheet. Make sure to take parts in solving the problems. Your participation score is a combination of being prepared, willing to explore the problem, working in groups and contributing toward the solution.

1. Let G^{-1} be the transformation defined by the equations

$$u = 8x - 2y \qquad v = 4x + 0.25y$$

(A) Find G by solving the system of equations for x and y in terms of u and v.

(B) Find the value of the Jacobian $\dfrac{\partial(x,y)}{\partial(u,v)}$.

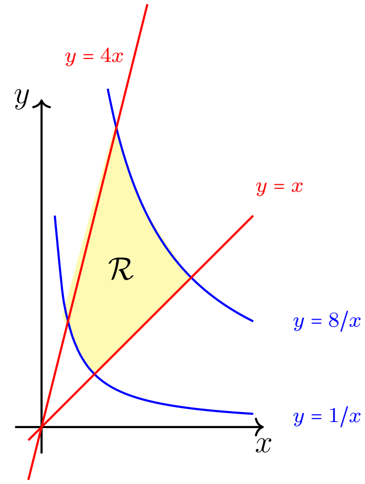
(C) Find the image under G^{-1} of the region in the xy-plane bounded by the x-axis, the y-axis, and $x + y = 1$. Sketch the transformed region in the uv-plane. [1]

https://youtu.be/z5z08DvmgIE

[1] Remember: for linear transformations, you can transform two points on a line segment and then draw the transformed line segment. "Linear transformations transform line to lines."

2. Let $\mathcal{R}$ be the region in the first quadrant bounded by $xy = 1$, $xy = 8$, $y = x$, and $y = 4x$.

(A) Find a transformation G such that the inverse region $G^{-1}(\mathcal{R})$ is rectangular.



(B) What is the $|\text{Jac}(G)|$?

(C) Set up $\iint_{\mathcal{R}} f(x, y)\, dA_{xy}$ using the change of variable in Part (A).

https://youtu.be/fsOIlfmwkSI and https://youtu.be/3ecpMeCWeuI

3. **Background Story:** First try a linear change of variables for triple integrals to transform the ellipsoid to a sphere. Then try a spherical transformation to compute the values for the sphere.

   (A) Find a change of variable that transforms the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ into the sphere

   $$u^2 + v^2 + w^2 = 1.$$

   (B) Find the Jacobian of the the transformation in Part (A).

   (C) Find the volume of the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ using the change of variable in Part (A).

4. **Background Story:** Remember: The polar, cylindrical and spherical conversions are transformations and for R be a region in $\mathbb{R}^3$ with coordinates x, y, z, if S be a region in $\mathbb{R}^3$ with coordinates u, v, w and G is a transformation that maps S to R:

$$\vec{\mathbf{G}}(u,v,w) = \big(x(u,v,w),\, y(u,v,w),\, z(u,v,w)\big).$$

Then

$$\iiint\limits_R f(x,y,z)\,dV_{xyz} = \iiint\limits_S f\big(\vec{\mathbf{G}}(u,v,w)\big)\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dV_{uvw}$$

where

$$\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = \left|\mathrm{Jac}(\vec{\mathbf{G}})\right| = abs\left(\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}\right).$$

Now let
$$\begin{aligned} (x,y) &= G(r,\theta) &&= (r\cos(\theta), r\sin(\theta)) && G:\mathbb{R}^2 \to \mathbb{R}^2 \\ (x,y,z) &= H(r,\theta,z) &&= (r\cos(\theta), r\sin(\theta), z) && H:\mathbb{R}^3 \to \mathbb{R}^3 \\ (x,y,z) &= I(\rho,\phi,\theta) &&= (\rho\sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta), \rho\cos(\phi)) && I:\mathbb{R}^3 \to \mathbb{R}^3 \end{aligned}$$

(A) Compute the Jacobian of polar transformation. $|\mathrm{Jac}(G)| =$

(B) Compute the Jacobian of cylindrical transformation. $|\mathrm{Jac}(H)| =$

(C) Compute the Jacobian of spherical transformation. $|\mathrm{Jac}(I)| =$

GroupWork Rubrics:

Preparedness: ——/0.5, Contribution: ——/0.5, Correct Answers: ——/0.5

Individual Portion of the Worksheet

Name: ________________

Upload this section individually on canvas or turn it in to your instructor on the 2^{nd} lab day of the week. You can ask questions in class and work in groups but you turn in the individual work. Start before the class so you can ask questions during the class. If you didn't complete the work in class, make sure to work on it outside the class and complete it. Show all your work; your score depends on the work you have shown.

5. **Background Story:** Sometimes a transformation can simplify the domain and simplify the integrand at the same time.

   Questions: *(3.5 points) Evaluate the integral using a linear change of variables.*

$$\iint_{\mathcal{R}} (x+y)e^{y^2-x^2}\,dA$$

   where $\mathcal{R}$ is the polygon with vertices $(6,0)$, $(0,6)$, $(-6,0)$, and $(0,-6)$.[2]

   Make sure to include:

   (A) A transformation or an inverse transformation, where the region transforms to a rectangular region.

   (B) A transformed rectangular region.

[2]Remember that for linear transformations, you can transform two points on a line segment and then draw the transformed line segment. Use this fact to draw the region.

(C) The Jacobian of the transformation.

(D) An iterated double integral where the bounds and the integrand have been converted.

(E) A final answer.

https://youtu.be/J5fAfbkoBjw and https://youtu.be/rL-bacMf4kE