Week 11-Lab 2: Worksheet 17: Sections 13.2-13.3

I said: "I talk about my privileges because you can find ways in my stories to create your own. You are in college, find mentors, sit in seminars and lectures. You can learn how to parse information. You can do it. You can excel."

Vector-Valued Functions

A **scalar function** is a function whose output is a scalar.

A **vector function** (or **vector-valued function**) is a function whose output is a vector.

Derivatives of Vector-Valued Functions

Therefore, the tangent line to the curve of $\vec{\mathbf{r}}(t)$ at $t = a$ can be parametrized by the vector function

$$\vec{l}(t) = \vec{\mathbf{r}}(a) + t\,\vec{\mathbf{r}}\,'(a).$$

For example, a vector function $\vec{\mathbf{r}} : \mathbb{R} \to \mathbb{R}^3$ has the form

$$\vec{\mathbf{r}}(t) = \langle f(t), g(t), h(t)\rangle = f(t)\vec{\mathbf{i}} + g(t)\vec{\mathbf{j}} + h(t)\vec{\mathbf{k}}$$

where the independent variable t is a scalar in $\mathbb{R}$ and the dependent variable $\vec{\mathbf{r}}(t)$ is a vector in $\mathbb{R}^3$. The scalar functions f, g, and h are the **components** of the vector function $\vec{\mathbf{r}}$.

Derivative of vector function $\vec{\mathbf{r}}(t) = \langle f(t), g(t), h(t)\rangle$ is a vector function $\vec{\mathbf{r}}'(t)$ whose components are derivatives of the component of the original function.

$$\vec{\mathbf{r}}'(t) = \langle f'(t), g'(t), h'(t)\rangle$$

Provided that $\vec{\mathbf{r}}\,'(t) \neq \vec{0}$, it is **tangent** to the curve parametrized by $\vec{\mathbf{r}}$.

Example: Parametrize the intersection of the surfaces $y^2 + z^2 = 4$ and $x = 5y^2$.

This example is explained in details here: https://www.geogebra.org/m/bbpeqwbn

Arc Length Formula The arc length of the curve parametrized by $\vec{\mathbf{r}}(t)$ for $a \le t \le b$ is

$$\int_a^b \|\vec{\mathbf{r}}\,'(t)\|\,dt = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}\,dt.$$

The Arc Length Function The length of the portion of the curve $\vec{\mathbf{r}}$ over the interval $[a, t]$ is

$$S(t) = \int_a^t \|\vec{\mathbf{r}}\,'(\tau)\|\,d\tau$$

The letter S is reserved for arc length. (Here τ is just a dummy variable.)
Note that $s(t)$ is a scalar function of t.

Arc Length Parametrization: ①Find the arc length functions. ②Solve for t in (the original parameter) in terms of S. ③ Replace the original parameter in the original parameterization by $t(s)$.

Group Work Portion of the Worksheet

Names: ____

Work in groups to do this portion of the worksheet. Make sure to take parts in solving the problems. Your participation score is a combination of being prepared, willing to explore the problem, working in groups and contributing toward the solution.

1. **Background Story:** In some cases finding intersection of two surfaces contains multiple pieces of disjoint connected smooth curves. (https://www.geogebra.org/m/zah9y6cu)

   Questions: Follow the following steps to find a vector equation for the tangent line to the curve of intersection of the surfaces at the point $(6,8,15)$.

$$x^2 + y^2 = 100 \qquad\qquad y^2 + z^2 = 289$$

   (A) Find a parametrization $\vec{\mathbf{r}}(t)$ of the curves of intersections that contains point $(6,8,15)$, $\mathcal{C}$.

   (B) What is the value t_0 such that $\vec{\mathbf{r}}(t_0) = (6,8,15)$?

   (C) Find a tangent vector to the curve $\mathcal{C}$ at $(6,8,15)$.

   (D) Find an equation for the tangent line at $(6,8,15)$.

   (E) Does $\mathcal{C}$ contain point $(6,8,-15)$?

2. Follow the steps to find an arc length parametrization of the given circle.

   Questions: Consider the circle in the plane $y = 8$ with radius 5 and center $(2, 8, 4)$.

   (A) Find a parametrization $\vec{\mathbf{r}}(t)$ of the circle.

   (B) Find the arclength function $s(t) = \int_0^t \|\vec{\mathbf{r}}\,'(\tau)\| d\tau$. (Note: I chose the lower limit of integration to be 0 in this case.)

   (C) Find inverse of $s(t)$.

   (D) Find $\vec{\mathbf{r}}\left(s^{-1}(t)\right)$ which is the arclength parametrization of the circle.

   (E) A circle can be presumed as the curve of an intersection of a cylinder and a plane. Give an example of such surfaces for this circle. (Note this part is not in the context of arclength parametrization.)

Video: https://youtu.be/a2cdLgQtF4c

3. Let $\vec{r}(t) = \langle t^2, t^3, t\rangle$ and $\vec{s}(t) = \langle e^{3t}, e^{2t}, e^t\rangle$, evaluate the following:

(A) Evaluate $\dfrac{d}{dt}\left(\vec{s}(t)\right)$.

(B) Evaluate $\dfrac{d}{dt}\left(\vec{r}\big(g(t)\big)\right)$ using the chain rule, where $g(t) = e^t$.

(C) $\dfrac{d}{dt}\left(\vec{r}(t)\cdot\vec{s}(t)\right)$

(D) $\dfrac{d}{dt}\left(\vec{r}(t)\times\vec{s}(t)\right)$

4. **Background Story:** Do outside the lecture. Curl of $\vec{\mathbf{F}}$ is

$$\operatorname{curl}(\vec{\mathbf{F}}) = \nabla \times \vec{\mathbf{F}} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \times \langle F_1, F_2, F_3 \rangle = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

or

$$\operatorname{curl}(\vec{\mathbf{F}}) = \left(\frac{\partial \vec{\mathbf{F}}_3}{\partial y} - \frac{\partial \vec{\mathbf{F}}_2}{\partial z} \right) \vec{\mathbf{i}} + \left(\frac{\partial \vec{\mathbf{F}}_1}{\partial z} - \frac{\partial \vec{\mathbf{F}}_3}{\partial x} \right) \vec{\mathbf{j}} + \left(\frac{\partial \vec{\mathbf{F}}_2}{\partial x} - \frac{\partial \vec{\mathbf{F}}_1}{\partial y} \right) \vec{\mathbf{k}}$$

Where $\vec{\mathbf{F}} = \langle \vec{\mathbf{F}}_1, \vec{\mathbf{F}}_2, \vec{\mathbf{F}}_3 \rangle$.

Questions: Show $\operatorname{curl}(\nabla f) = \vec{0}$ when $f(x, y, z)$ is an arbitrary scalar function with continuous second derivatives.

5. **Background Story:** Do before the lecture. Div of $\vec{\mathbf{F}}$ is

$$\operatorname{div}(\vec{\mathbf{F}}) = \nabla \cdot \vec{\mathbf{F}} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle F_1, F_2, F_3 \rangle = \frac{\partial \vec{\mathbf{F}}_1}{\partial x} + \frac{\partial \vec{\mathbf{F}}_2}{\partial y}, \frac{\partial \vec{\mathbf{F}}_3}{\partial z}$$

Questions: Show $\operatorname{div}(\operatorname{curl}\vec{\mathbf{F}}) = 0$ when $\vec{\mathbf{F}}(x, y, z)$ is an arbitrary vector function with continuous second derivatives.

GroupWork Rubrics:

Preparedness: ___/0.5, Contribution: ___/0.5, Correct Answers: ___/0.5

Individual Portion of the Worksheet

Name: ________________

Upload this section individually on canvas or turn it in to your instructor on the 2nd lab day of the week. You can ask questions in class and work in groups but you turn in the individual work. Start before the class so you can ask questions during the class. If you didn't complete the work in class, make sure to work on it outside the class and complete it. Show all your work; your score depends on the work you have shown.

Both individual questions of this week are posted in Worksheet 16.