Week 12-Lab 2: Worksheet 18: Section 16.1

Vincent asked me to tell you: "Don't worry! You already have won!"

Vector Fields: **Definition:** A **vector field** in $\mathbb{R}^n$ is a function $\vec{\mathbf{F}} : \mathbb{R}^n \to \mathbb{R}^n$.

The Del Operator (Formula for Curl, Divergence and Gradient):

The **del** or **nabla** operator[1] ∇ is defined by $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$.

Applying ∇ to a scalar function f gives its gradient:

$$\text{Gradient:} \qquad \nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

The curl and divergence of a vector field can also be written in terms of ∇:

$$\text{Divergence:} \qquad \operatorname{div}(\vec{\mathbf{F}}) = \nabla \cdot \vec{\mathbf{F}} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle F_1, F_2, F_3 \rangle = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$\text{Curl:} \qquad \operatorname{curl}(\vec{\mathbf{F}}) = \nabla \times \vec{\mathbf{F}} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \times \langle F_1, F_2, F_3 \rangle = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

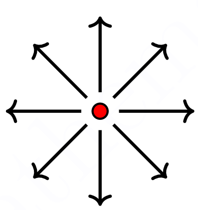
A few properties:

$\nabla \times (\nabla f) = \vec{0}$ or curl of gradient is zero.

$\nabla \cdot (\nabla \times \vec{\mathbf{F}}) = 0$ or divergence of the curl $= 0$.

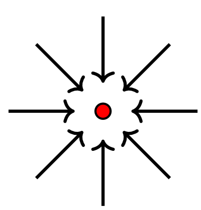
Physical properties of Div., Grad. and Curl:

Divergence of a vector field: The divergence of a vector field $\vec{\mathbf{F}}$ at a point P measures how much $\vec{\mathbf{F}}$ disperses "stuff" near P. $\operatorname{Div}(\vec{\mathbf{F}})$ is a scalar function.



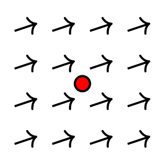
$\operatorname{div}(\vec{\mathbf{F}}) > 0$

disperses stuff (source)



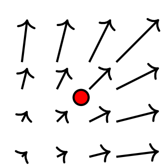
$\operatorname{div}(\vec{\mathbf{F}}) < 0$

attracts stuff (sink)



$\operatorname{div}(\vec{\mathbf{F}}) = 0$

"incompressible"

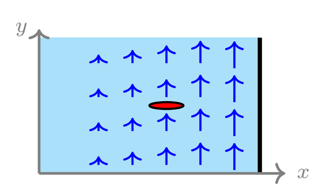


$\operatorname{div}(\vec{\mathbf{F}}) > 0$

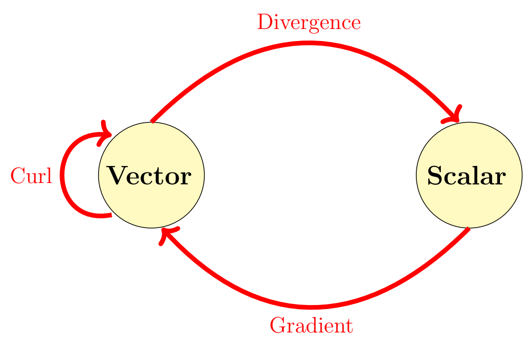
Curl of a vector field:

The **curl** of a vector field $\vec{\mathbf{F}}$ measures how $\vec{\mathbf{F}}$ causes objects to rotate.

Example: The current in a river is stronger near the banks than in the middle. A boat is anchored near the right bank. What happens to the boat? **It rotates counterclockwise.**



$\text{Curl}_z(\vec{\mathbf{F}}(x,y)) > 0$ is a vector field sticking out of the page toward us.



The mnemonic is provided by Trevor Arrigoni.

Curl	Cross Product
Divergence	Dot Product

The mnemonic is provided by Ryan Frier.

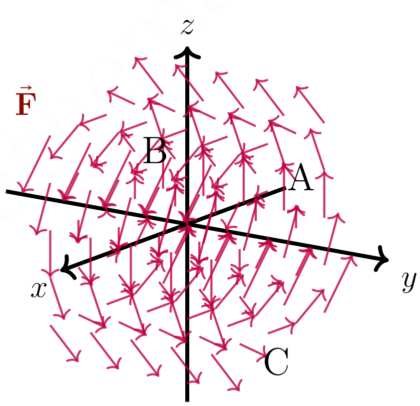
Group Work Portion of the Worksheet

Names: ____

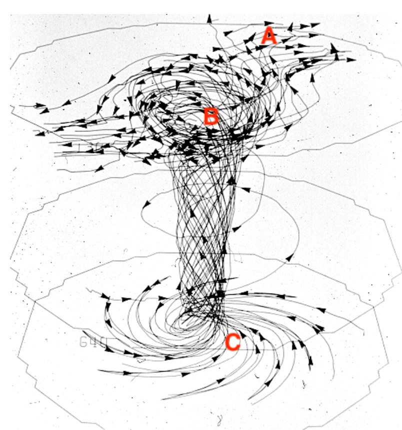
Work in groups to do this portion of the worksheet. Make sure to take parts in solving the problems. Your participation score is a combination of being prepared, willing to explore the problem, working in groups and contributing toward the solution.

1. If you have not done so already, in Worksheet 17, show that $\text{Div}(\text{Curl}(\vec{\mathbf{F}})) = 0$ and $\text{Curl}(\nabla f) = \vec{0}$, on your own.

2. Use the right hand rule and draw a vector in the direction of curl at points A, B and C for each vector field.



https://www.geogebra.org/m/xgssdqwj



Vector Field $\vec{\mathbf{T}}$: Airflow of a Tornado

3. Let $f(x,y,z)$ be a scalar function and $\vec{\mathbf{F}}(x,y,z)$ be a vector field. Fill in the blank with either "a vector field" or "a scalar function". *(Use the last mnemonic on the previous page for this problem.)*

   (A) $\nabla f(x,y,z)$ is _______________. That is, gradient of $f(x,y,z)$ is a _______________.

   (B) $\nabla \cdot \vec{\mathbf{F}}(x,y,z)$ is _______________. That is, divergence of $\vec{\mathbf{F}}(x,y,z)$ is a _______________.

   (C) $\nabla \times \vec{\mathbf{F}}(x,y,z)$ is _______________. That is, curl of $\vec{\mathbf{F}}(x,y,z)$ is a _______________.

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4. **Background Story:** Let's discuss a vector field with many applications in physics and electromagnetism.

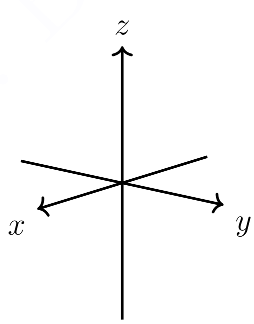
   Questions:

   (A) Let be the scalar-valued function of three variables $r = \sqrt{x^2 + y^2 + z^2}$. Find ∇r.

   (B) Find the domain of ∇r.

   (C) Evaluate $\|\nabla r\|$.

   (D) Remember for a point $P = (x,y,z)$, the position vector is $\vec{\mathbf{r}} = \langle x,y,z\rangle$. Denote $\vec{e}_r = \dfrac{\vec{\mathbf{r}}}{\|\vec{\mathbf{r}}\|}$. Graph the vector field $\vec{e}_r$ on it's domain. (Draw two vectors in each octant.) [2]



   (E) A scalar potential for this vector field is a function $f(x,y,z)$ such that $\nabla f(x,y,z) = \vec{\mathbf{F}}(x,y,z)$. Find a scalar potential function for $\vec{e}_r$? (Hint: You know this from previous parts.)

   (F) Find $\text{curl}(\vec{e}_r)$.

   (G) Find $\text{div}(\vec{e}_r)$.

[2] Vector fields $\vec{\mathbb{E}} = k\vec{e}_r$, where k is a constant, have many applications in physics and will be denoted differently in different area of science. This exercise helps you know their scalar potential function and properties.

5. **Background Story:** Come up with the steps on your own. Video: https://youtu.be/Eq9JezYXZRQ

Questions: Let $\vec{F}(x,y,z) = \langle z\cos(y), -xz\sin(y) + e^y, x\cos(y)\rangle$.

(A) Show that Curl($\vec{\mathbf{F}}$) = $\vec{0}$.

(B) Verify that all components of $\vec{\mathbf{F}}$ are continuous.

(C) The conditions in Parts (A) and (B) are enough to ensure that $\vec{\mathbf{F}}$ is conservative. That is $\vec{\mathbf{F}} = \nabla f$, where $f(x,y,z)$ is a scalar potential. That $\langle F_1,\ F_2\ , F_3\rangle = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\rangle$. Find the antiderivatives to find the :

Find $\int F_1\, dx =$ ______________ , $\int F_2\, dy =$ ______________ , $\int F_3\, dz =$ ______________ .

(D) Find the potential function f and verify that $\nabla f = \vec{\mathbf{F}}$.

GroupWork Rubrics day 2:

Preparedness: ——/0.5, Contribution: ——/0.5, Correct Answers: ——/0.5 GroupWork Rubrics:

Preparedness: ——/0.5, Contribution: ——/0.5, Correct Answers: ——/0.5

Individual Portion of the Worksheet

Name: ________________

Upload this section individually on canvas or turn it in to your instructor on the 2$^{\text{nd}}$ lab day of the week. You can ask questions in class and work in groups but you turn in the individual work. Start before the class so you can ask questions during the class. If you didn't complete the work in class, make sure to work on it outside the class and complete it. Show all your work; your score depends on the work you have shown.

This week, there is no individual portion. Upload your **practice exam** instead. No additional work is needed.