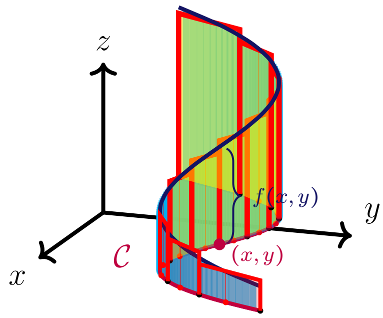
Week 13-Lab 1: Worksheet 19: Section 16.2

Ryan quoted: "In academia, you have a lot of freedom in your schedule. You may have to work 7 days a week but you get to pick which 7 days you work." I reminded myself not to make you workaholics as well.

Scalar Line Integral:



If $\mathcal{C}$ is a smooth curve in $\mathbb{R}^2$ parametrized by a function $\vec{\mathbf{r}}(t)$, and f is continuous on $\mathcal{C}$, then

$$\underbrace{\int_{\mathcal{C}} f(x,y)\,ds}_{\text{Notation}} = \underbrace{\int_a^b f\big(\vec{\mathbf{r}}(t)\big)\,\|\vec{\mathbf{r}}\,'(t)\|\,dt.}_{\text{Formula}}$$

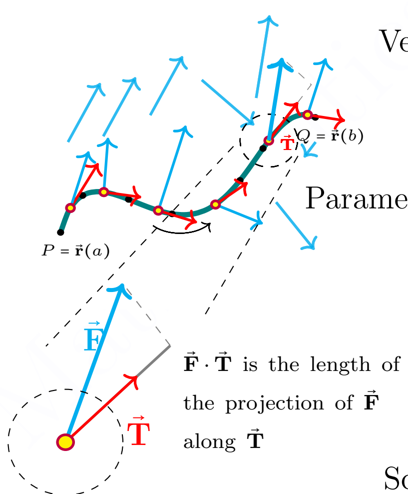
The same formula works for curves in $\mathbb{R}^n$ (for $n = 2, 3, \ldots$):

$$\int_{\mathcal{C}} f\,ds = \int_{\mathcal{C}} f(x_1, \ldots, x_n)\,ds = \int_a^b f\big(\vec{\mathbf{r}}(t)\big)\,\|\vec{\mathbf{r}}\,'(t)\|\,dt$$

The symbol $ds = \|\vec{\mathbf{r}}\,'(t)\|\,dt$ is called the **arc length element**. It represents a little bit of the arc length of the curve.

Vector Line Integral:

Let $\vec{\mathbf{F}}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle$ be a vector field and $\mathcal{C}$ a curve parametrized by $\vec{\mathbf{r}}(t) = \langle x(t), y(t), z(t)\rangle$ for $[a,b]$.



Vector Differential Form: $\underbrace{\int_{\mathcal{C}} \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}}}_{\text{Notation}} = \underbrace{\int_{\mathcal{C}} \vec{\mathbf{F}}\cdot\vec{\mathbf{T}}\,ds}_{\text{Physical Property}}$

Parametric Vector Evaluation: $\underbrace{\int_a^b \vec{\mathbf{F}}\big(x(t), y(t), z(t)\big)\cdot\vec{\mathbf{r}}\,'(t)\,dt}_{\text{A formula}}$

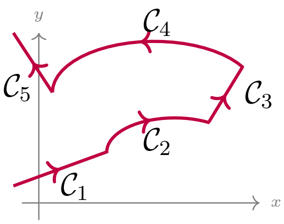
$= \underbrace{\int_a^b \vec{\mathbf{F}}\big(\vec{\mathbf{r}}(t)\big)\cdot\vec{\mathbf{r}}\,'(t)\,dt}_{\text{Another format of the same formula}}$

Scalar Differential Form: $\int_{\mathcal{C}} P\,dx + Q\,dy + R\,dz$

Parametric Scalar Evaluation:

$$\int_a^b (P(x(t), y(t), z(t))x'(t) + Q(x(t), y(t), z(t))y'(t) + R(x(t), y(t), z(t))z'(t))dt$$

Piecewise-Smooth Curves



$\mathcal{C}$ is **piecewise-smooth** if $\mathcal{C}$ is the union of a finite number of smooth curves $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n$.

In that case,

$$\int_{\mathcal{C}} f\,ds = \int_{\mathcal{C}_1} f\,ds + \int_{\mathcal{C}_2} f\,ds + \cdots + \int_{\mathcal{C}_n} f\,ds$$

and

$$\int_{\mathcal{C}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_{\mathcal{C}_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}_1 + \int_{\mathcal{C}_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}_2 + \cdots + \int_{\mathcal{C}_n} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}_n$$

Group Work Portion of the Worksheet

Names: ____

Work in groups to do this portion of the worksheet. Make sure to take parts in solving the problems. Your participation score is a combination of being prepared, willing to explore the problem, working in groups and contributing toward the solution.

1. **Background Story:** Take 2-3 minutes to discuss this problem with your team. Don't write anything down. (This is already on the lecture slides.)

   (A) Let $\vec{\mathbf{F}}(\vec{\mathbf{r}}(a))$ be a force at point $\vec{\mathbf{r}}(a)$ on smooth parametrized path $\mathcal{C} : \vec{\mathbf{r}}(t)$. Let $\vec{\mathbf{T}}(a) = \dfrac{\vec{\mathbf{r}}\,'(a)}{\|\vec{\mathbf{r}}\,'(a)\|}$ be the unit tangent vector to $\mathcal{C}$ at point $\vec{\mathbf{r}}(a)$. What is the physical interpretation of $\vec{\mathbf{F}}(\vec{\mathbf{r}}(a)) \cdot \vec{\mathbf{T}}(a)$?

   (B) Let $\vec{\mathbf{F}} : \mathbb{R}^n \to \mathbb{R}^n$ be a force field (a vector field representing a force at each point of space). Let s be the arclength parameter for smooth parametrized path $\mathcal{C}:\ \vec{\mathbf{r}}(t)$. What is the physical interpretation of quantity $(\vec{\mathbf{F}} \cdot \vec{\mathbf{T}})\Delta s$ at each point $\vec{\mathbf{r}}(a)$?

   (C) Using the assumptions in Part (B), what is the physical interpretation of $\int_{\mathcal{C}} \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\ ds$?

2. **Background Story:** Follow the instructions in the footnote. Specially learn how to parametrize a line segment. Write a lot, this problem is one of the essentials for your muscle memory.

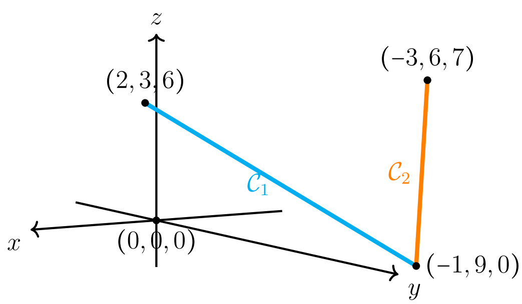
   Questions: Define $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_1$ where $\mathcal{C}_1$ is the line segment from $(2,3,6)$ to $(-1,9,0)$, and $\mathcal{C}_2$ is the line segment from $(-1,9,0)$ to $(-3,6,7)$.

   (A) Discuss, among your team, why $\cup$ is used in $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$ for 1-2 minutes.

   (B) Set up the following **scalar line integral** as a single variable integral. **Do not** evaluate.

$$\int_{\mathcal{C}} xyz^2\, ds$$

[1]



Step 1: $\mathcal{C}_1 :\ \vec{\mathbf{r}}_1(t) = \langle \quad , \quad , \quad \rangle$

$\le t \le$

Step 2: $\vec{\mathbf{r}}_1\,'(t) = \langle \quad , \quad , \quad \rangle$

$\|\vec{\mathbf{r}}_1\,'(t)\| =$

Step 3: $f(\vec{\mathbf{r}}_1(t)) =$

Step 4: $\int_a^b f(\vec{\mathbf{r}}_1(t))\, \|\vec{\mathbf{r}}_1\,'(t)\| dt =$

Step 1: $\mathcal{C}_2 :\ \vec{\mathbf{r}}_2(t) = \langle \quad , \quad , \quad \rangle$

$\le t \le$

Step 2: $\vec{\mathbf{r}}_2\,'(t) = \langle \quad , \quad , \quad \rangle$

$\|\vec{\mathbf{r}}_2\,'(t)\| =$

Step 3: $f(\vec{\mathbf{r}}_2(t)) =$

Step 4: $\int_a^b f(\vec{\mathbf{r}}_2(t))\, \|\vec{\mathbf{r}}_2\,'(t)\| dt =$

$\int_{\mathcal{C}} f(x,y,z) ds =$

---

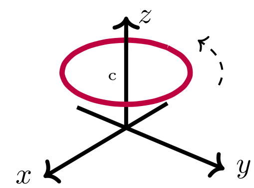
[1] Note:

- A very useful parameterization of a line segment from point (a,b,c) to (d,e,f) is $\vec{\mathbf{r}}(t) = \langle a,b,c\rangle + t\,\langle d-a, e-b, f-c\rangle$ where $0 \le t \le 1$.
- Parameterize each line segment $\mathcal{C}_1$ and $\mathcal{C}_1$ and $\mathcal{C}_2$, with $\vec{\mathbf{r}}_1(t)$ and $\vec{\mathbf{r}}_2(t)$; where each $\vec{\mathbf{r}}_i$ is the parametrization obtained by the above method.
- Then compute $\|\vec{\mathbf{r}}\,_i'(t)\|$ and compute $\int_{\mathcal{C}_i} f(x,y,z)ds = \int_0^1 f(\vec{\mathbf{r}}_i(t))\, \|\vec{\mathbf{r}}\,_i'(t)\|\, dt$ for each $i = 1,2$.
- Compute $\int_{\mathcal{C}} f(x,y,z)ds = \int_{\mathcal{C}_1} f(x,y,z)ds + \int_{\mathcal{C}_2} f(x,y,z)ds$.
- Video: https://mediahub.ku.edu/media/t/1_naubd1qy

3. **Background Story:** Compare scalar line integral with vector line integral. Write a lot, this problem is one of the essentials for your muscle memory.

Questions: Let $\mathcal{C}$ be the circle of radius 5 in plane $z = 4$ and center at $(0,0,4)$ in the direction shown.

(A) Evaluate $\displaystyle\int_{\mathcal{C}} \underbrace{3x^2z}_{f(x,y,z)}\, ds.$



Step 1: $\vec{\mathbf{r}}(t) = \langle \qquad , \qquad , \qquad \rangle$

$\leq t \leq$

Step 2: $\vec{\mathbf{r}}\,'(t) = \langle \qquad , \qquad , \qquad \rangle$

$\|\vec{\mathbf{r}}\,'(t)\| =$

Step 3: $f(\vec{\mathbf{r}}(t)) =$

Step 4: $\displaystyle\int_a^b f(\vec{\mathbf{r}}(t))\, \|\vec{\mathbf{r}}\,'(t)\| dt =$

(B) Evaluate $\displaystyle\int_{\mathcal{C}} \underbrace{\langle 0, xz, 5y\rangle}_{\vec{\mathbf{F}}(x,y,z)} \cdot d\vec{\mathbf{r}}.$

Step 1: $\vec{\mathbf{r}}(t) = \langle \qquad , \qquad , \qquad \rangle$

$\leq t \leq$

Step 2: $\vec{\mathbf{r}}\,'(t) = \langle \qquad , \qquad , \qquad \rangle$

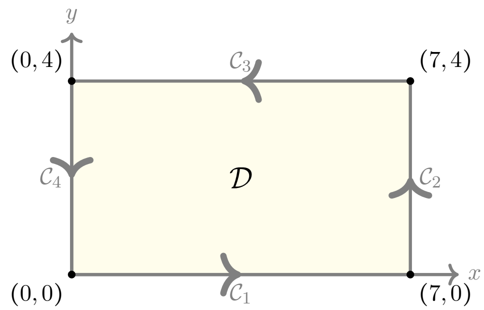
Step 3: $\vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) =$

Step 4: $\vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{r}}\,'(t) =$

Step 5: $\displaystyle\int_a^b \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{r}}\,'(t)\; dt =$

An example of vector line integral: https://mediahub.ku.edu/media/t/1_4zaolv67

4. **Background Story:** Do before lecture 17.1. The following process is to verify the Green's theorem for $\vec{\mathbf{F}}(x,y) = \langle xy, x-y\rangle$ over the region $\mathcal{D}$ whose boundary is the rectangle with vertices $(0,0)$, $(7,0)$, $(7,4)$, $(0,4)$ by following answering Items (A)-(D).



Questions:

(A) Parameterize each line segment. $\mathcal{C}_1$: $\vec{\mathbf{r}}_1(t)$ the line segment from $(0,0)$ to $(7,0)$, $\mathcal{C}_2$: $\vec{\mathbf{r}}_2(t)$ the line segment from $(7,0)$ to $(7,4)$, $\mathcal{C}_3$: $\vec{\mathbf{r}}_3(t)$ the line segment from $(7,4)$ to $(0,4)$, and $\mathcal{C}_4$: $\vec{\mathbf{r}}_4(t)$ the line segment from $(0,4)$ to $(0,0)$. (As shown in the picture.)

(B) Compute each $\int_{\mathcal{C}_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, $\int_{\mathcal{C}_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, $\int_{\mathcal{C}_3} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ and $\int_{\mathcal{C}_4} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$.

(C) Let $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{C}_4$. Use Part (B) to compute $\int_{\mathcal{C}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \oint_{\mathcal{C}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$.

(D) Compute $\iint_{\mathcal{D}} \operatorname{curl}(\vec{\mathbf{F}}) \cdot \vec{\mathbf{k}}\, dA$.

(E) Are the values in Parts (C) and (D) equal?

GroupWork Rubrics:

Preparedness: ——/0.5, Contribution: ——/0.5, Correct Answers: ——/0.5

Individual Portion of the Worksheet

Name: ________________

Upload this section individually on canvas or turn it in to your instructor on the 2nd lab day of the week. You can ask questions in class and work in groups but you turn in the individual work. Start before the class so you can ask questions during the class. If you didn't complete the work in class, make sure to work on it outside the class and complete it. Show all your work; your score depends on the work you have shown.

5. Let $\vec{\mathbf{F}}(x,y,z) = \langle 2xy, x^2 - 7z, -7y + 12 \rangle$.

   (A) *(0.5 points) Calculate curl*$(\vec{\mathbf{F}})$.

   (B) (0.5 points) Is $\vec{\mathbf{F}}$ *conservative? Justify your answer.*

   (C) (2.5 points) If it is conservative, then find a potential function. If it is not conservative, explain why.

6. **Background Story:** Please read the instructions in the footnote.

   Questions: *(3.5 points)* A 150 pound person carries a 20 pound can of paint up a helical staircase that encircles a tower with a radius of 28 feet. The tower is 180 feet high and makes exactly three complete revolutions. If there is a hole in the can of paint and 12 pounds of paint leaks steadily out of the can during the person's ascent, how much work is done by the person against gravity in climbing the stairs? [2]

[2] *Steps:*

- *Parametrize the staircase using the radius, the height and the number of rotations. (Refer to Section 13.3 lecture notes for an example.)*
- *Compute the vector field of downward gravitational force (Parallel to z-axis) at any point on the staircase. Note that the weight is decreasing over time in a linear fashion.*
- *Note that work is computed using the* **vector** *line integral.*