# Week 13-Lab 2: Worksheet 20: Section 16.2-16.3

*Ryan quoted: "In academia, you have a lot of freedom in your schedule. You may have to work 7 days a week but you get to pick which 7 days you work." I reminded myself not to make you workaholics as well.*

## Conservative Vector Fields:

Let $f(x, y, z)$ be a scalar-valued function. Its gradient is a vector field:

$$\vec{\mathbf{F}} = \nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

The function $f$ is called a **(scalar) potential function** for $\vec{\mathbf{F}}$.

A vector field is called **conservative** if it has a potential function,

**Theorem:** If $\vec{\mathbf{F}}$ is conservative on an open connected domain $\mathcal{R}$, then any two potential functions of $\vec{\mathbf{F}}$ differ by a constant.

**Theorem:** If $\vec{\mathbf{F}}$ is a conservative vector field in $\mathbb{R}^2$ or $\mathbb{R}^3$, then $\text{Curl}(\vec{\mathbf{F}}) = \vec{0}$.

**Theorem:** A vector field $\vec{\mathbf{F}}$ on an open connected domain $\mathcal{D}$ is path-independent if and only if it is conservative.

**Theorem:** Let $\vec{\mathbf{F}}$ be a vector field on a **simply connected** domain $\mathcal{D}$.
If $\text{Curl}(\vec{\mathbf{F}}) = \vec{0}$ in $\mathcal{D}$, then $\vec{\mathbf{F}}$ is conservative.

**Finding Scalar Potentials** The process for finding scalar potential functions is essentially antidifferentiation, but with a twist.

**For $\vec{\mathbf{F}}(x, y) = \langle F_1(x, y),\ F_2(x, y)\rangle$:**

1. Find the indefinite integrals $\int F_1(x, y)\, dx$ and $\int F_2(x, y)\, dy$.
   - **The constants of integration are $c_1(y)$ and $c_2(x)$ respectively** (instead of the usual "$+C$"), because if $\frac{\partial}{\partial x}(f(x, y)) = F_1$ then $\frac{\partial}{\partial x}(f(x, y) + c_1(y)) = F_1$ as well.
2. "Match up the pieces" to determine $f(x, y)$.

**For $\vec{\mathbf{F}}(x, y, z) = \langle F_1(x, y, z),\ F_2(x, y, z),\ F_3(x, y, z)\rangle$:**

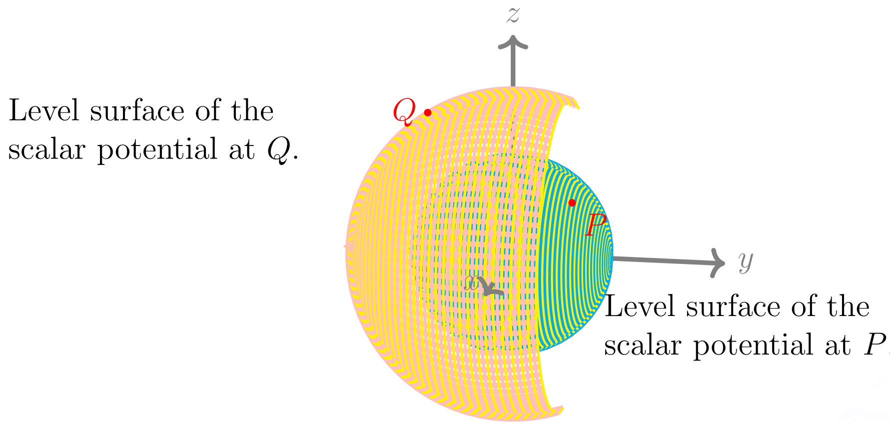
1. Find the indefinite integrals $\int F_1\, dx$, $\int F_2\, dy$, and $\int F_3\, dz$.
   Constants of integration: $c_1(y, z)$, $c_2(x, z)$, $c_3(x, y)$.
2. "Match up the pieces" to determine $f(x, y, z)$.

## Fundamental Theorem for Conservative Vector Fields

Assume that $\vec{\mathbf{F}} = \nabla f$ on a connected domain $\mathcal{D}$. If $\vec{\mathbf{r}}$ is a path along a curve $\mathcal{C}$ from $P$ to $Q$ in $\mathcal{D}$, then

$$\int_{\mathcal{C}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = f(Q) - f(P).$$

Note: The property in this theorem is also called Path Independence.




## Parametrization of Line Segment

A very useful parameterization of a line segment from point $(a, b, c)$ to $(d, e, f)$ is $\vec{\mathbf{r}}(t) = \langle a, b, c \rangle + t \langle d - a, e - b, f - c \rangle$ where $0 \le t \le 1$.

In this case, $\vec{\mathbf{r}}'(t) = \langle d - a, e - b, f - c \rangle$.

# Group Work Portion of the Worksheet

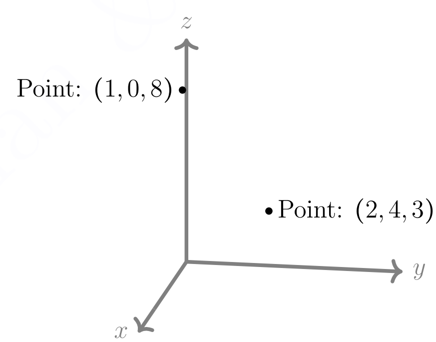
**Names:** ____________________________________________

Work in groups to do this portion of the worksheet. Make sure to take parts in solving the problems. Your participation score is a combination of being prepared, willing to explore the problem, working in groups and contributing toward the solution.

1. Remember the gradient (conservative) field $\vec{e}_r(x, y, z) = \dfrac{\vec{\mathbf{r}}}{\|\vec{\mathbf{r}}\|}$ from Worksheet 18. We showed that $r = \sqrt{x^2 + y^2 + z^2}$ was a **scalar potential** for the $\vec{e}_r$.

   (A) Describe and draw the level surfaces for $r$ at $(2, 4, 3)$ and $(1, 0, 8)$.

   (B) Draw the vector field $\vec{e}_r$ at points $(2, 4, 3)$ and $(1, 0, 8)$.




   (C) Explain how the vector field is related to the level surfaces.

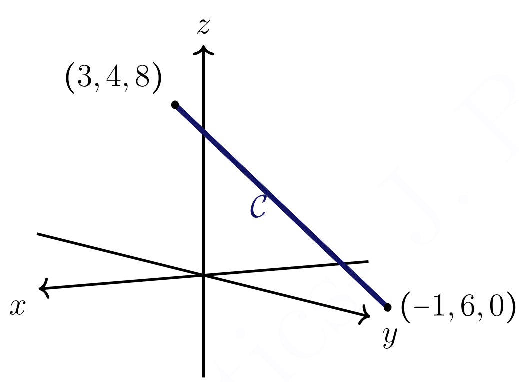
   (D) Compute $\int_{\mathcal{C}} \vec{e}_r \cdot d\vec{\mathbf{r}}$ when $\mathcal{C}$ is any path from $(2, 4, 3)$ to $(1, 0, 8)$.

2. The force exerted by an electric charge at the origin on a charged particle at a point $(x, y, z)$ with position vector $\vec{\mathbf{r}} = \langle x, y, z \rangle$ is $\vec{\mathbf{E}}(\vec{\mathbf{r}}) = K\dfrac{\vec{\mathbf{r}}}{\|\vec{\mathbf{r}}\|^3}$ where $K$ is a constant.

(A) Find a **scalar potential function** for $\vec{\mathbf{E}}$.

(B) Is the domain simply connected?

(C) Use the potential function to find the work done as the particle moves along a straight line from $(-1, 6, 0)$ to $(3, 4, 8)$.




(D) Parametrize $\mathcal{C}$, the line segment from $(3, 4, 8)$ to $(-1, 6, 0)$ with that orientation, and set up $\int_{\mathcal{C}} \vec{\mathbf{E}} \cdot d\vec{\mathbf{r}}$ but do **NOT** evaluate. (We want you to realize how difficult the computation is when the Theorem of Conservative Vector Fields is not used.)

3. Consider the vector field $\vec{F}(x,y,z) = \langle \cos(z), -9, -x\sin(z) \rangle$.

(A) Is $\vec{F}$ is **conservative** on $\mathbb{R}^3$? (Hint: Show both conditions are satisfied.)

(B) Find the **scalar potential** function $f$ for the gradient field $\vec{F}$.

(C) Evaluate $\int_C \vec{F} \cdot d\vec{r}$ along $C$ given by $\vec{r}(t) = (e^t, e^{2t}, t)$ from point $(1,1,0)$ to $(e^\pi, e^{2\pi}, \pi)$.

Video: https://youtu.be/4_wrYIL-Ocw

GroupWork Rubrics:

Preparedness: ——/0.5, Contribution: ——/0.5, Correct Answers: ——/0.5

## Individual Portion of the Worksheet

**Name:** ________________

Upload this section individually on canvas or turn it in to your instructor on the 2nd lab day of the week. You can ask questions in class and work in groups but you turn in the individual work. Start before the class so you can ask questions during the class. If you didn't complete the work in class, make sure to work on it outside the class and complete it. Show all your work; your score depends on the work you have shown.

Both individual questions are given in Worksheet 19.