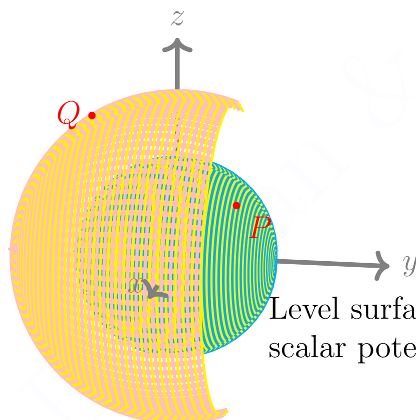
Week 14-Lab 1: Worksheet 21: Section 17.1

I asked: " What is a simple closed curve?" They showed me an example: " ♡ " I said: " Aah! This is a simple closed curve but it is not a smooth curve!"

Fundamental Theorems of Line Integrals:

Section 16.3: Fundamental Theorem for Conservative Vector Fields: Assume that $\vec{\mathbf{F}} = \nabla f$ on a domain $\mathcal{D}$. For *any* curve $\mathcal{C}$ from P to Q in $\mathcal{D}$,

$$\int_{\mathcal{C}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = f(Q) - f(P).$$

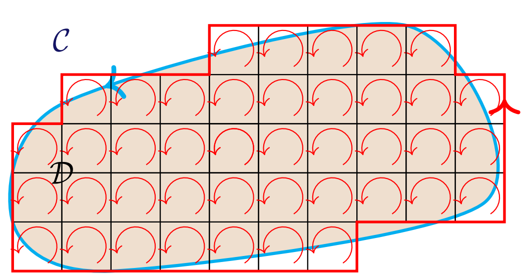


Definition: A curve is *simple* if it does not intersect itself. It is *closed* if it begins and ends at the same point. A parameterization of a simple, closed curve is *positively oriented* if the point moves counterclockwise.

- **Section 17.1: Green's Theorem:** If $\mathcal{D}$ is a domain whose boundary $\partial\mathcal{D}$ is a simple, closed curve with positive orientation, then

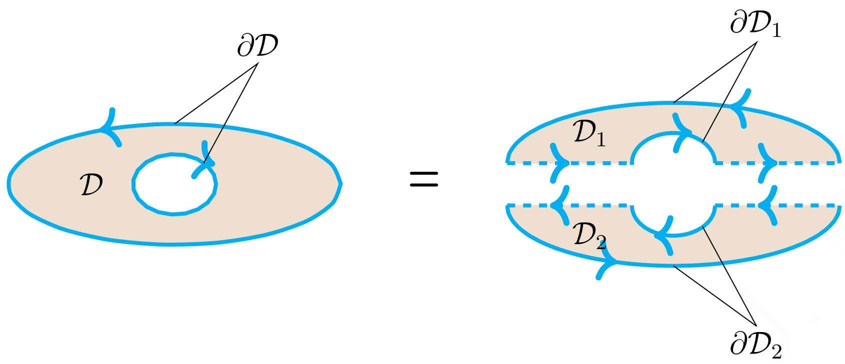
$$\int_{\partial\mathcal{D}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_{\partial\mathcal{D}} P\,dx + Q\,dy = \iint_{\mathcal{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_{\mathcal{D}} \mathrm{Curl}(\vec{\mathbf{F}})_z\,dA$$

The *boundary* of a surface $\mathcal{S}$ is denoted $\partial\mathcal{S}$. When $\mathcal{S}$ is oriented, the induced boundary orientation is the direction which keeps the surface on the left if you were to walk along the boundary with your feet on the curve and your head pointed in the direction of the orientation of the surface.



Regions With Holes: For a connected region with holes, the boundary consists of two or more closed curves. Every part of the boundary must be oriented to keep the region on the **left**.

Outside boundary: counterclockwise. Inside boundary: clockwise.



Two Ways we Use Green's Theorem in Calculus III:

1. When finding a contour vector line integral is complicated but finding the double integral of $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial x}$ on the region is easy. That is, you calculate the right hand side to find the left hand side.
2. When finding the area of a region is complicated but computing the vector line integral is possible. That is, you calculate the left hand side to find the right hand side.

The type of Problems to Expect in This Section:

1. Verification of the theorem by computing both the line integral and the double integral.
2. Using the double integral to compute the line integral.
3. Finding an appropriate vector field vector field where curl is one. For example, $\vec{\mathbf{F}} = \langle -y/2, x/2\rangle$, $\vec{\mathbf{G}} = \langle 0, x\rangle$, and $\vec{\mathbf{H}} = \langle -y, 0\rangle$. Then computing the line integral which then will be equal to the enclosed area.
4. Regions with holes where "outside" boundaries go with + if they are oriented counterclockwise and − if otherwise; the "inside" boundaries go with + if they are oriented clockwise and − otherwise.

We also discuss the vortex vector field where the curl is zero but the field is not conservative. This is not a typical example but a good counter example.

Group Work Portion of the Worksheet

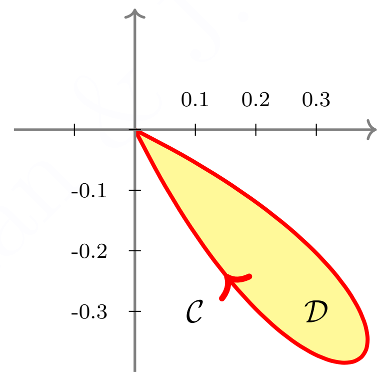
Names: ____

Work in groups to do this portion of the worksheet. Make sure to take parts in solving the problems. Your participation score is a combination of being prepared, willing to explore the problem, working in groups and contributing toward the solution.

1. **Background Story:** This question serves two purposes: ① how to find the the direction of the curve if the parameterization is given and ② how to use Green's Theorem to find the area.

   Let $\mathcal{C}$ be the simply closed curve defined by

   $$\mathcal{C} : \vec{r}(t) = \langle t(t-1)(t-2),\ t(t-1)(t+1)\rangle \text{ for } 0 \le t \le 1.$$



   (A) Find $\vec{\mathbf{r}}(0)$, $\vec{\mathbf{r}}(0.25)$ and $\vec{\mathbf{r}}(0.5)$. Then confirm that the parameterization transverses the curve in clockwise orientation.

   (B) For $\vec{F}(x,y) = \langle 0, x\rangle$, compute $\oint_{\partial D} \vec{F} \cdot d\vec{r}$.[1]

   (C) For $\vec{F}(x,y) = \langle 0, x\rangle$, find $\text{curl}(\vec{\mathbf{F}})$; use the value to simplify $\iint_{\mathcal{D}} \text{curl}_z(\vec{\mathbf{F}})\, dA$.[2]

   (D) Use Green's Theorem [3] to find the area entrapped in the simple closed curve $\mathcal{C}$.

   Video: https://mediahub.ku.edu/media/t/1_lmoyu2t7

[1] Take the line integral of $\vec{\mathbf{F}}$ on $\vec{\mathbf{r}}(t)$ for $0 \le t \le 1$; then multiply by -1 since $\vec{\mathbf{r}}(t)$ is in clockwise direction.

[2] Note $\vec{\mathbf{F}}$ is a placeholder vector field and many vector fields exists with the property that $\text{curl}_z(\vec{\mathbf{F}}) = 1$.

[3] $\oint_{\mathcal{C}} \vec{F} \cdot d\vec{r} = \iint_{\mathcal{D}} \text{curl}_z(\vec{\mathbf{F}})\, dA$

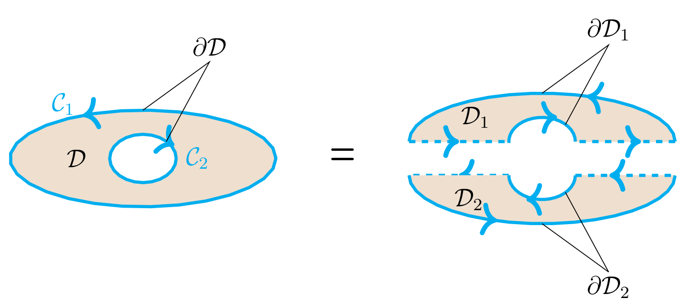
2. **Background Story:** Let's discuss Green's Theorem for regions with holes. We start with the region $\mathcal{D}$, on the left, and we **cut** it into two regions $\mathcal{D}_1$ and $\mathcal{D}_2$, on the right. Region $\mathcal{D}$ has outer boundary $\mathcal{C}_1$ in counterclockwise orientation and inner boundary $\mathcal{C}_2$ in clockwise orientation.

   Discuss the following questions in 3-4 minutes but do not write anything down.

   (A) Explain why $\int_{\partial\mathcal{D}_1} \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}} + \int_{\partial\mathcal{D}_2} \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}}$ is equal to $\int_{\mathcal{C}_1} \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}} + \int_{\mathcal{C}_2} \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}}$. *(Make sure to mention the line integral on the line of the cut.)*

   (B) Why is $\iint_{\mathcal{D}} \text{Curl}_z(\vec{\mathbf{F}})\, dA = \iint_{\mathcal{D}_1} \text{Curl}_z(\vec{\mathbf{F}})\, dA + \iint_{\mathcal{D}_2} \text{Curl}_z(\vec{\mathbf{F}})\, dA$ true?

   (C) Using Parts (A) and (B), explain why $\int_{\mathcal{C}_1} \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}} + \int_{\mathcal{C}_2} \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}} = \iint_{\mathcal{D}} \text{Curl}_z(\vec{\mathbf{F}})\, dA$.



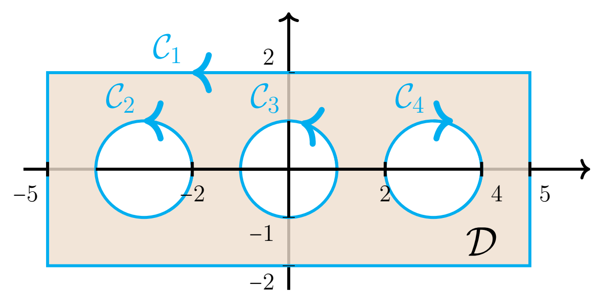
3. Consider the following region with three holes, $\mathcal{D}$. Let $\operatorname{curl}\left(\vec{F}\right)_z = -4$ and

$$\int_{\mathcal{C}_2} \vec{F} \cdot d\vec{r} = 7 \qquad \int_{\mathcal{C}_3} \vec{F} \cdot d\vec{r} = 6 \qquad \int_{\mathcal{C}_4} \vec{F} \cdot d\vec{r} = 3.$$

(A) Write the line integral over the boundary of $\mathcal{C} = \partial\mathcal{D}$ as a sum or difference of $\oint_{\mathcal{C}_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, $\oint_{\mathcal{C}_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, $\oint_{\mathcal{C}_3} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, and $\oint_{\mathcal{C}_4} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$.[4]

(B) Use Part (A) to write the generalized statement of Green's theorem.

(C) Use Part (B) to compute $\oint_{\mathcal{C}_1} \vec{F} \cdot d\vec{r}$.



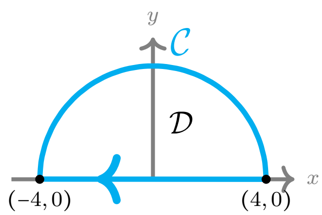
Similar Video: https://mediahub.ku.edu/media/t/1_4bj90g7k

[4] Remember in Green's Theorem's with holes, each piece must be oriented to keep the region on the **left**.
• Outside boundary: counterclockwise. • Inside boundary: clockwise.

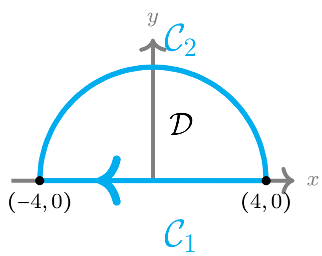
4. **Background Story:** If curl($\vec{\mathbf{F}}$) is easy to compute; computing a contour vector line integral by Green's Theorem is advised.

   Questions: A particle starts at the point $(4,0)$, moves along the x-axis to $(-4,0)$, and the along the semicircle to $y = \sqrt{16-x^2}$ back to the starting point.

   (A) How much work was done by the vector field $\vec{F}(x,y) = \langle -5x^2y, 5xy^2 + \ln(y^2+1)\rangle$ in moving the particle along the path.



   (B) How much work was done by the vector field $\vec{F}(x,y) = \langle -5x^2y, 5xy^2 + \ln(y^2+1)\rangle$ in moving the particle along the upper circle path $\mathcal{C}_2$ only.



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5. **Background Story:** In Section 16.3, we discuss the **vortex vector field** and then we discussed it again in Section 17.1 in regions with holes. In Section 17.1, we showed an easy way to compute all contour integrals for the vortex field. We said if the curve goes around origin counterclockwise once, the contour integral is 2π. If the curve does not go around origin at all, the contour integral is 0. This is a good example where having simply connected domains is one of the sufficient conditions for conservative fields.

Example 3 from the lecture, Section 16.3: Is $\vec{\mathbf{F}}(x,y) = \left\langle \dfrac{-y}{x^2+y^2}, \dfrac{x}{x^2+y^2} \right\rangle$ conservative?

<u>Answer:</u> **No,** because it is not path-independent. If $\mathcal{C}_1$ is the unit circle, with standard parametrization $\vec{\mathbf{r}}_1(t) = \langle \cos(t), \sin(t) \rangle$ for $0 \le t \le 2\pi$, then

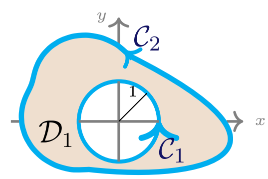
$$\oint_{\mathcal{C}_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_0^{2\pi} \overbrace{\langle -\sin(t), \cos(t) \rangle}^{\vec{\mathbf{F}}(\vec{\mathbf{r}}_1(t))} \cdot \overbrace{\langle -\sin(t), \cos(t) \rangle}^{\vec{\mathbf{r}}_1{}'(t)} dt = \int_0^{2\pi} dt = 2\pi.$$

$$\text{On the other hand, } \frac{dF_1}{dy} = \frac{dF_2}{dx} = \frac{y^2 - x^2}{(x^2+y^2)^2} \quad \therefore \quad \text{Curl}(\vec{\mathbf{F}}) = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & 0 \end{vmatrix} = \vec{0}.$$

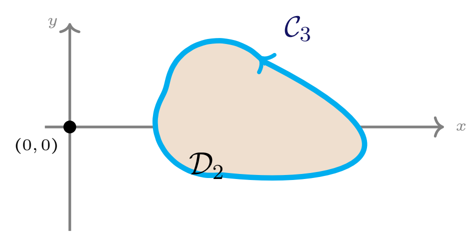
What is going on here? The domain of $\vec{\mathbf{F}}$ is $\mathbb{R}^2 \setminus \{(0,0)\}$, which is not simply connected!

Questions:

(A) Use Green's Theorem for regions with holes and the curl computed above to compute $\oint_{\mathcal{C}_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$.



(B) If $\mathcal{C}_3$ boundary of an arbitrary simply-connected region that does not contain the origin, as shown to the right. Compute $\oint_{\mathcal{C}_3} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ using Green's Theorem.



GroupWork Rubrics:

Preparedness: ——/0.5, Contribution: ——/0.5, Correct Answers: ——/0.5

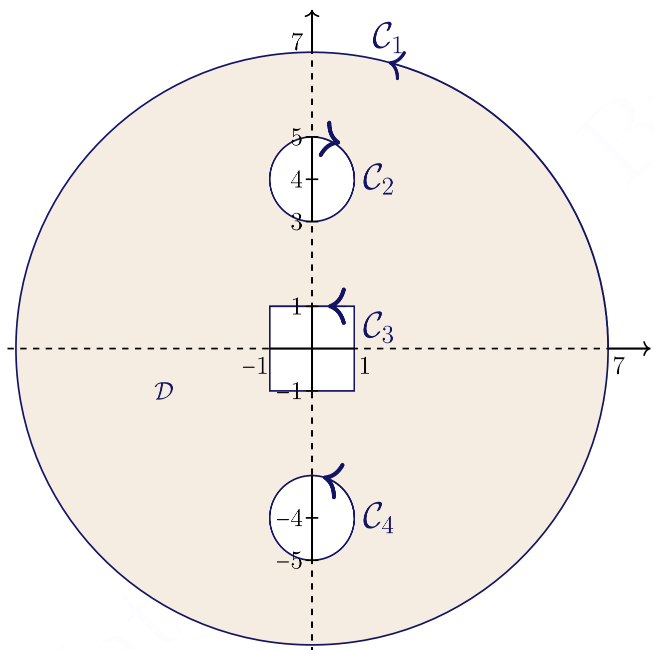
Individual Portion of the Worksheet

Name: ________________

Upload this section individually on canvas or turn it in to your instructor on the 2nd lab day of the week. You can ask questions in class and work in groups but you turn in the individual work. Start before the class so you can ask questions during the class. If you didn't complete the work in class, make sure to work on it outside the class and complete it. Show all your work; your score depends on the work you have shown.

6. *(3.5 points)* Find $\oint_{\mathcal{C}_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ if $\operatorname{curl}\big(\vec{\mathbf{F}}\big)_z = 6$ in the region defined by the 4 curves and

$$\oint_{\mathcal{C}_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = 3 \qquad \oint_{\mathcal{C}_3} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = 7 \qquad \oint_{\mathcal{C}_4} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \pi$$



7. Let $\mathcal{C}$ consists of the arc of the curve piece-wise smooth curve $\mathcal{C}$ that is the union of $\mathcal{C}_1 : y = \sin(x)$ from $(0,0)$ to $(\pi,0)$ and $\mathcal{C}_2$: the line segment from $(\pi,0)$ to $(0,0)$.
Let $\vec{\mathbf{F}}(x,y,z) = \left\langle \sqrt{x} + 5, x^2 + \sqrt{y} \right\rangle$.

(A) *(0.5 points)* Compute $\mathrm{Curl}_z(\vec{\mathbf{F}})$.

(B) *(1.25 points)* Evaluate $\oint_{\mathcal{C}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$.

(C) *(1.75 points)* Evaluate $\int_{\mathcal{C}_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$.