Week 15-Lab 1: Worksheet 22 Sections 16.4-16.5

I said: "The semester is near its end. The light at the end of the tunnel is brighter. Keep going to lectures and labs. Let's finish strong!"

Scalar Surface Integrals

Suppose that $f(x, y, z)$ is a function on a surface $\mathcal{S}$ parametrized by $\vec{\mathbf{G}}(u, v)$ over the domain $\mathcal{R}$.

The **surface integral** of f over $\mathcal{S}$ is defined as

$$\iint_{\mathcal{S}} f(x, y, z)\, dS = \iint_{\mathcal{R}} f\left(\vec{\mathbf{G}}(u, v)\right) \|\vec{\mathbf{G}}_u \times \vec{\mathbf{G}}_v\|\, dA$$

The symbol dS is called the **surface element** or the **area element**:

$$dS = \|\vec{\mathbf{G}}_u \times \vec{\mathbf{G}}_v\|\, dA_{uv}.$$

Vector Surface Integrals

If $\vec{\mathbf{F}}$ is a continuous vector field defined on an oriented surface $\mathcal{S}$ with unit normal vector $\vec{\mathbf{n}}$, then the **vector surface integral** of $\vec{\mathbf{F}}$ over $\mathcal{S}$ is

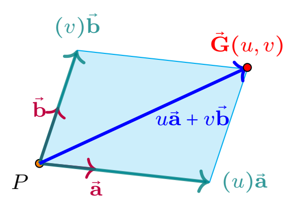
$$\iint_{\mathcal{S}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iint_{\mathcal{S}} \vec{\mathbf{F}} \cdot \vec{\mathbf{n}}\, dS.$$

The integral is also called the **flux** of $\vec{\mathbf{F}}$ across $\mathcal{S}$.

A Parametric Plane

The **plane** containing the point P and the vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ can be parametrized by

$$\vec{\mathbf{G}}(u, v) = \vec{OP} + u\vec{\mathbf{a}} + v\vec{\mathbf{b}}.$$



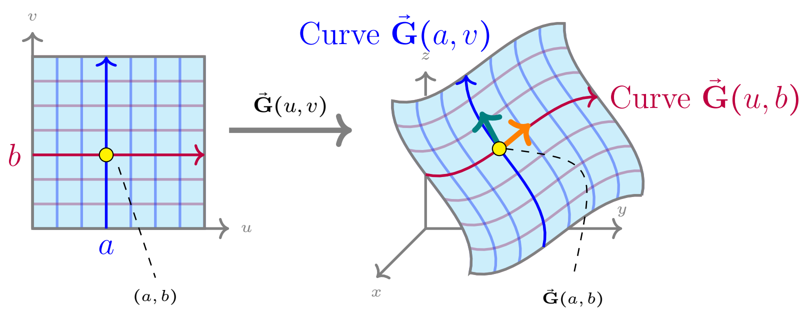
<u>Idea:</u> Every point in the plane can be obtained by starting at P and moving parallel to the vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$.

Now a parametrization of a parallelogram whose two adjacent sides at point P are $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ is

$$\vec{\mathbf{G}}(u, v) = \vec{OP} + u\vec{\mathbf{a}} + v\vec{\mathbf{b}} \text{ where } 0 \le u \le 1 \text{ and } 0 \le v \le 1.$$

Surface Integrals: (Sections 16.4 and 16.5)

A curve is *smooth* if it has a parametrization $\vec{\mathbf{r}}(t)$ where $\vec{\mathbf{r}}\,'$ is continuous. A parametrization is *regular* if $\vec{\mathbf{r}}(t)\,' \ne \vec{0}$ for any t.



A surface is *smooth* if it has a parametrization $\vec{\mathbf{G}}(u, v)$ where $\vec{\mathbf{G}}_u \times \vec{\mathbf{G}}_v$ is continuous. A parametrization is *regular* if $\vec{\mathbf{G}}_u \times \vec{\mathbf{G}}_v$ is nonzero.

Let f be a scalar function and $\vec{\mathbf{F}}$ a vector field.

Scalar Line Integral along a smooth curve $\mathcal{C}$ with a regular parametrization $\vec{\mathbf{r}}\,(t)$ on $[a, b]$.

$$\int_{\mathcal{C}} f\, ds = \int_a^b f(\vec{\mathbf{r}}\,(t))\, \|\vec{\mathbf{r}}\,'(t)\|\, dt$$

If $f = 1$ then $\int_{\mathcal{C}} f\, ds$ is the arclength of $\mathcal{C}$.

Vector Line Integral, or work done by a vector field, along an oriented curve $\mathcal{C}$:

$$\int_{\mathcal{C}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_a^b \vec{\mathbf{F}}\,(\vec{\mathbf{r}}\,(t)) \cdot \vec{\mathbf{r}}\,'(t)\, dt$$

Scalar Surface Integral over a smooth surface $\mathcal{S}$ with a regular parametrization $\vec{\mathbf{G}}(u, v)$ on $\mathcal{R}$:

$$\iint_{\mathcal{S}} f\, dS = \iint_{\mathcal{R}} f(\vec{\mathbf{G}}(u, v))\, \|\vec{\mathbf{G}}_u \times \vec{\mathbf{G}}_v\|\, dA$$

If $f = 1$ then $\iint_{\mathcal{S}} f\, dS$ is the surface area of $\mathcal{S}$.

Vector Surface Integral or **flux** of a vector field $\vec{\mathbf{F}}$ through an oriented surface $\mathcal{S}$:

$$\iint_{\mathcal{S}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iint_{\mathcal{R}} \vec{\mathbf{F}}\left(\vec{\mathbf{G}}(u, v)\right) \cdot \underbrace{\left(\pm \vec{\mathbf{G}}_u \times \vec{\mathbf{G}}_v\right)}_{\vec{\mathbf{N}}} dA$$

Group Work Portion of the Worksheet

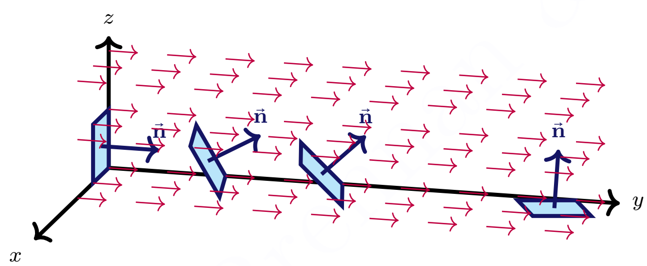
Names: ____

Work in groups to do this portion of the worksheet. Make sure to take parts in solving the problems. Your participation score is a combination of being prepared, willing to explore the problem, working in groups and contributing toward the solution.

1. **Background Story:** Flux is the flow of a flow field through a surface. Take 2-5 minute to discuss these questions.

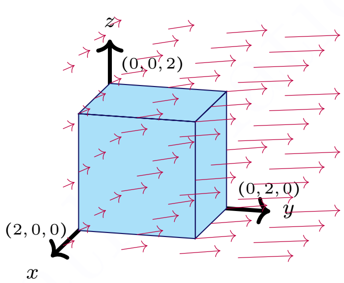
   Questions:

   (A) Through which surface is the flux of $\vec{\mathbf{F}}(x,y,z) = \langle 0, 0.5, 0\rangle$ largest?



   (B) What type of integral measures the flow out of a surface?

   (C) Let's say $\text{Div}(\vec{\mathbf{F}})$ is the rate at which the dispersing density changes at any point in the space; flux through a surface is the flow of matter through the surface. How do you think the outward net flow of the matter through surface of this box is related to $\text{Div}(\vec{\mathbf{F}})$?



   (D) If the $\text{Div}(\vec{\mathbf{F}}) = 5$ in the above figure. What do you think is the flux out of the box?

2. **Background Story:** To find the Cartesian equations, eliminate the parameters between x, y and z. Note that in the case of trig functions, solving for $\sin(w)$, $\cos(w)$ and then using Pythagorean Theorem may be a proffered method.

   Questions: Use the elimination stated in each part to identify each surface with the given parameterization.

   (A) Eliminate v between x and y. Then eliminate v between the answer and z.

   $$\vec{A}\,(u,v) = \langle u+v, 5-v, 1+7u+3v\rangle$$

   > **Solution:** Since
   > $\vec{A}(u,v) = \langle 0,5,1\rangle + u\,\langle 1,0,7\rangle + v\,\langle 1,-1,3\rangle$,
   > $\vec{A}$ is a **plane** through the point $(0,5,1)$ containing the vectors $\langle 1,0,7\rangle$ and $\langle 1,-1,3\rangle$.
   > Or by eliminating u and v, the plane
   > $z = 1 + 7\,(x+y-5) + 3\,(5-y)$

   (B) Eliminate u between x and y.

   $$\vec{B}\,(u,v) = \langle 2\sin(u), 3\cos(u), v\rangle$$

   (C) Eliminate u and v between all three variables at the same time.

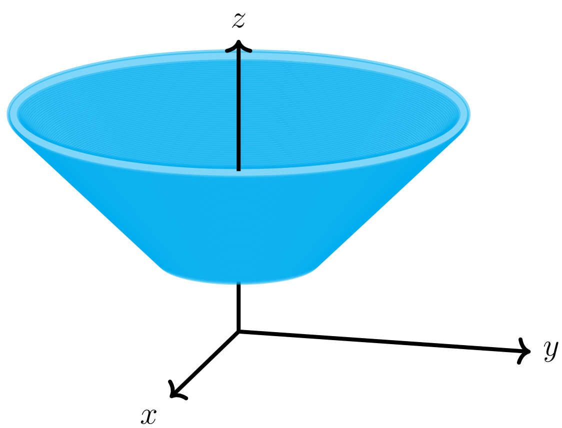
   $$\vec{C}\,(u,v) = \left\langle u, v, u^2 - v^2\right\rangle$$

   (D) Decide how to go about the eliminations.

   $$\vec{D}\,(u,v) = \left\langle u\sin(4\,v), u^2, u\cos(4\,v)\right\rangle$$

3. **Background Story:** This question is a scalar surface integral. (Section 16.4) Your hint should be that the function, ρ, is a scalar function.

Questions: Find the mass of a thin funnel in the shape of a cone $z = \sqrt{x^2+y^2}$, $1 \le z \le 7$, if its density function is $\rho(x,y,z) = 49 - (x^2+y^2)$ kg/unit.



Step 1: $\vec{\mathbf{G}}(r,\theta) = \langle \quad , \quad , \quad \rangle$

$\le r \le$ and $\le \theta \le$

Step 2: $\vec{\mathbf{G}}_r \times \vec{\mathbf{G}}_\theta = \langle \quad , \quad , \quad \rangle$

Step 3: $\|\vec{\mathbf{G}}_r \times \vec{\mathbf{G}}_\theta\| =$

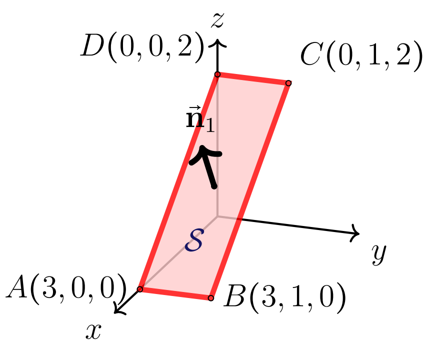
Step 4: $\rho(\vec{\mathbf{G}}(r,\theta)) =$

Step 5: $\iint_{\mathcal{D}} \rho(\vec{\mathbf{G}}(r,\theta))\|\vec{\mathbf{G}}_r \times \vec{\mathbf{G}}_\theta\| \, dA_{r\theta} =$

4. **Background Story:** The purpose of this problem is ① Parameterizing rectangles. ② Comparing the orientation of $\vec{\mathbf{N}}$ and $\vec{\mathbf{n}}$ using the geometry. ③ Understanding that if $\vec{\mathbf{n}}$ and $\vec{\mathbf{N}}$ have opposite orientation, it is not a big deal and you multiply $\vec{\mathbf{N}}$ by a negative sign. ④ You see how the grid curves are incorporated in surface elements.

Questions:

(A) Parameterize the following rectangular region using the formula: $\vec{\mathbf{G}}(u,v) = \overrightarrow{OA} + \overrightarrow{AB}\,u + \overrightarrow{AD}\,v$, for $0 \le u, v \le 1$.



(B) Compute $\vec{\mathbf{G}}_u$, $\vec{\mathbf{G}}_v$ and $\vec{\mathbf{N}} = \vec{\mathbf{G}}_u \times \vec{\mathbf{G}}_v$ for Part (A). Is $\vec{\mathbf{N}}$ oriented in the same direction as $\vec{\mathbf{n}}_1$ or in the opposite direction?[1]

(C) Let $\vec{\mathbf{F}}(x,y,z) = \langle 3x, 4y, 5\rangle$. Parameterize $\vec{\mathbf{F}}$ on the surface and compute integral

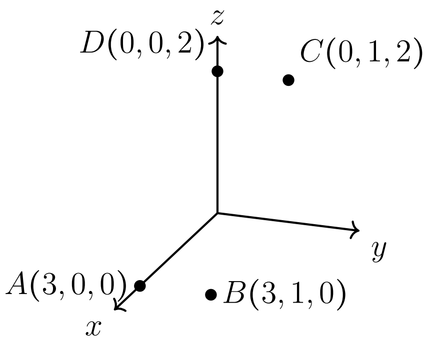
$$\int_0^1 \int_0^1 \vec{\mathbf{F}}(\vec{\mathbf{G}}(u,v)) \cdot \vec{\mathbf{N}}\, du\, dv.$$

[2]

[1] You can draw the vector you found and compare.

[2] This integral is the flux through the $\mathcal{S}$, assuming the orientation of $\vec{\mathbf{N}}$.

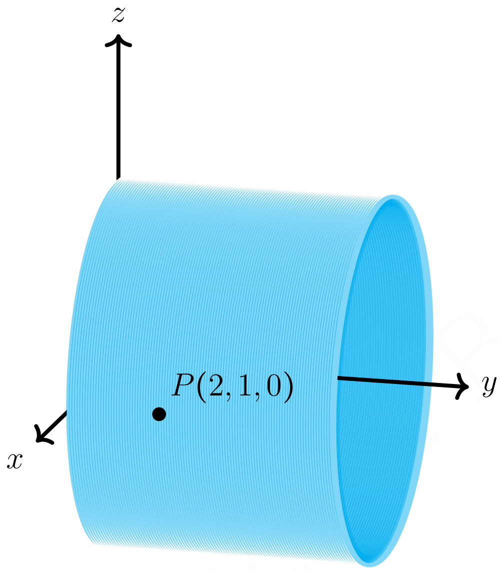
(D) Draw the region parameterized by $\vec{\mathbf{H}}(u,v) = \overrightarrow{OA} + \overrightarrow{AB}\,u + \overrightarrow{AC}\,v$, for $0 \le u, v \le 1$.[3]



(E) Let $\vec{\mathbf{I}}(u,v) = \langle 2\cos(u), v, 2\sin(u)\rangle$ for $0 \le u \le 2\pi$ and $0 \le v \le 3$ be a parameterization of cylinder $x^2 + z^2 = 4$ and $0 \le y \le 3$.

(i) Draw the curves $\vec{\mathbf{I}}(2\pi, v)$ and $\vec{\mathbf{I}}(u,1)$ going through $P(2,1,0) = \vec{\mathbf{I}}(2\pi,1)$.

(ii) Draw $\vec{\mathbf{I}}_u(2\pi,1)$ and $\vec{\mathbf{I}}_v(2\pi,1)$ at point $P = \vec{\mathbf{I}}(2\pi,1)$.



(iii) Find

(i) $\vec{\mathbf{I}}_u \times \vec{\mathbf{I}}_v$

(ii) $\|\vec{\mathbf{I}}_u \times \vec{\mathbf{I}}_v\|$

(iii) $\vec{\mathbf{n}}_3 = \dfrac{\vec{\mathbf{I}}_u \times \vec{\mathbf{I}}_v}{\|\vec{\mathbf{I}}_u \times \vec{\mathbf{I}}_v\|}$

(iv) Draw or describe $\vec{\mathbf{n}}_3$ at few points on the surface.

[3] Check your answers by entering the parameterization in Geogebra sheet: https://www.geogebra.org/m/qsmkvcmq

GroupWork Rubrics:

Preparedness: ——/0.5, Contribution: ——/0.5, Correct Answers: ——/0.5

Individual Portion of the Worksheet

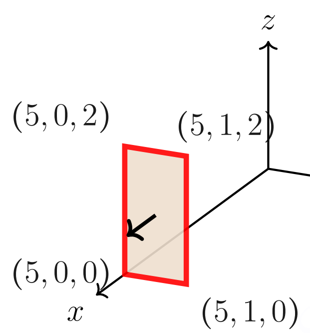
Name: ________________

Upload this section individually on canvas or turn it in to your instructor on the 2nd lab day of the week. You can ask questions in class and work in groups but you turn in the individual work. Start before the class so you can ask questions during the class. If you didn't complete the work in class, make sure to work on it outside the class and complete it. Show all your work; your score depends on the work you have shown.

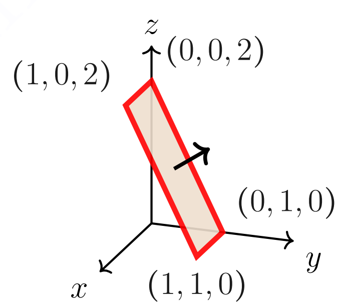
5. Let $\vec{\mathbf{F}} = \langle 4z^2, 5x, 3y \rangle$.

   (A) *(0.5 points) Compute* $curl(\vec{\mathbf{F}})$.

   (B) (2.5 points) Parameterize each rectangular region and compute the flux $\iint_{\mathcal{S}} curl(\vec{\mathbf{F}}) \cdot d\vec{\mathbf{S}}$ *through surface of each of the rectangular regions below, assuming each is oriented as shown.*



(i)



(ii)

(i)

Step 1: $\vec{\mathbf{G}}(u,v) = \langle \quad , \quad , \quad \rangle$

$\leq u \leq$ *and* $\leq v \leq$

Step 2: $\vec{\mathbf{G}}_u \times \vec{\mathbf{G}}_v = \langle \quad , \quad , \quad \rangle$

$\vec{\mathbf{N}} = \langle \quad , \quad , \quad \rangle$

Step 3: $Curl(\vec{\mathbf{F}}(\vec{\mathbf{G}}(u,v))) = \langle \quad , \quad , \quad \rangle$

Step 4: $\iint_{\mathcal{D}} Curl(\vec{\mathbf{F}}(\vec{\mathbf{G}}(u,v))) \cdot \vec{\mathbf{N}}\, dA_{uv} =$

(ii)

Step 1: $\vec{\mathbf{G}}(u,v) = \langle \quad , \quad , \quad \rangle$

$\leq u \leq$ *and* $\leq v \leq$

Step 2: $\vec{\mathbf{G}}_u \times \vec{\mathbf{G}}_v = \langle \quad , \quad , \quad \rangle$

$\vec{\mathbf{N}} = \langle \quad , \quad , \quad \rangle$

Step 3: $Curl(\vec{\mathbf{F}}(\vec{\mathbf{G}}(u,v))) = \langle \quad , \quad , \quad \rangle$

Step 4: $\iint_{\mathcal{D}} Curl(\vec{\mathbf{F}}(\vec{\mathbf{G}}(u,v))) \cdot \vec{\mathbf{N}}\, dA_{uv} =$

   (C) (0.5 points) Draw the direction on the boundary of each surface, $\mathcal{S}$*, so that the flux computed is equal to* $\oint_{\partial\mathcal{S}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$.[4]

[4] Use Stokes' Theorem. $\oint_{\partial\mathcal{S}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_{\mathcal{S}} \text{curl}(\vec{\mathbf{F}}) \cdot d\vec{\mathbf{S}}$

6. **Background Story:** This is an example of two different parameterization of surfaces and how they work with surface integration.

Questions: This question is regarding parametrization of a disk, $\mathcal{D}$, cut off from plane $z = 7 - x$ by $x^2 + y^2 = 16$ and $\vec{\mathbf{F}}(x, y, z) = \langle x^2, 0, y^2 \rangle$.

(A) *(1 point)* Find the $\vec{\mathbf{G}}_u \times \vec{\mathbf{G}}_v$ for $\vec{\mathbf{G}}(u, v) = \langle u\cos(v), u\sin(v), 7 - u\cos(v) \rangle$ for $0 \le u \le 4$ and $0 \le v \le 2\pi$.

(B) *(1 point)* Compute the $\iint_{\mathcal{D}} \vec{\mathbf{F}}(\vec{\mathbf{G}}(u, v)) \cdot d\vec{\mathbf{S}}$, orientation of $\vec{\mathbf{n}}$ is shown in the figure.

(C) *(1 point)* Compute $\vec{\mathbf{H}}_x \times \vec{\mathbf{H}}_y$ for $\vec{\mathbf{H}}(x, y) = \langle x, y, 7 - x \rangle$ for $-\sqrt{16 - x^2} \le y \le \sqrt{16 - x^2}$ (or $x^2 + y^2 \le 16$).

(D) *(0.5 points)* Find the $\iint_{\mathcal{D}} \vec{\mathbf{F}}(\vec{\mathbf{H}}(u, v)) \cdot d\vec{\mathbf{S}}$, orientation of $\vec{\mathbf{n}}$ is shown in the figure.