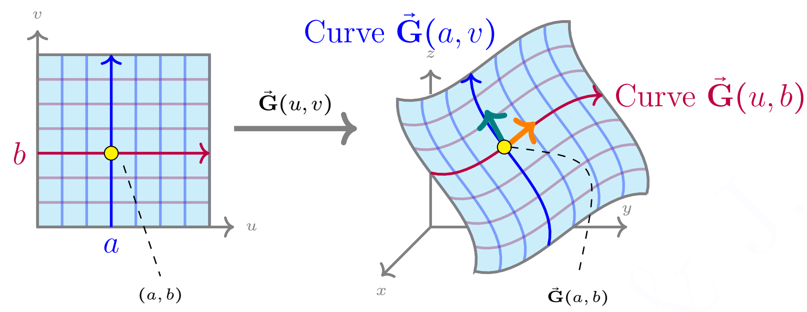
Week 15-Lab 2: Worksheet 23: Sections 16.4-16.5,17.2,17.3

Surface Integrals: (Sections 16.4 and 16.5)

A curve is *smooth* if it has a parametrization $\vec{\mathbf{r}}(t)$ where $\vec{\mathbf{r}}\,'$ is continuous. A parametrization is *regular* if $\vec{\mathbf{r}}(t)\,' \neq \vec{0}$ for any t.



A surface is *smooth* if it has a parametrization $\vec{\mathbf{G}}(u,v)$ where $\vec{\mathbf{G}}_u \times \vec{\mathbf{G}}_v$ is continuous. A parametrization is *regular* if $\vec{\mathbf{G}}_u \times \vec{\mathbf{G}}_v$ is nonzero.

Let f be a scalar function and $\vec{\mathbf{F}}$ a vector field.

Scalar Line Integral along a smooth curve $\mathcal{C}$ with a regular parametrization $\vec{\mathbf{r}}\,(t)$ on $[a,b]$.

$$\int_{\mathcal{C}} f\,ds = \int_a^b f(\vec{\mathbf{r}}\,(t))\,\|\vec{\mathbf{r}}\,'(t)\|\,dt$$

If $f = 1$ then $\int_{\mathcal{C}} f\,ds$ is the arclength of $\mathcal{C}$.

Vector Line Integral, or work done by a vector field, along an oriented curve $\mathcal{C}$:

$$\int_{\mathcal{C}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_a^b \vec{\mathbf{F}}\,(\vec{\mathbf{r}}\,(t)) \cdot \vec{\mathbf{r}}\,'(t)\,dt$$

Scalar Surface Integral over a smooth surface $\mathcal{S}$ with a regular parametrization $\vec{\mathbf{G}}(u,v)$ on $\mathcal{R}$:

$$\iint_{\mathcal{S}} f\,dS = \iint_{\mathcal{R}} f(\vec{\mathbf{G}}(u,v))\,\|\vec{\mathbf{G}}_u \times \vec{\mathbf{G}}_v\|\,dA$$

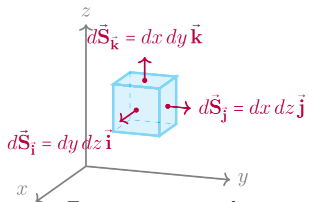
If $f = 1$ then $\iint_{\mathcal{S}} f\,dS$ is the surface area of $\mathcal{S}$.

Vector Surface Integral or **flux** of a vector field $\vec{\mathbf{F}}$ through an oriented surface $\mathcal{S}$:

$$\iint_{\mathcal{S}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iint_{\mathcal{R}} \vec{\mathbf{F}}\left(\vec{\mathbf{G}}(u,v)\right) \cdot \underbrace{\left(\pm\vec{\mathbf{G}}_u \times \vec{\mathbf{G}}_v\right)}_{\vec{\mathbf{N}}}\,dA$$

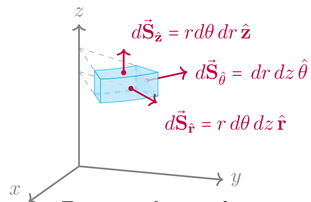
Memorization Method for Surface Elements:

Cartesian:



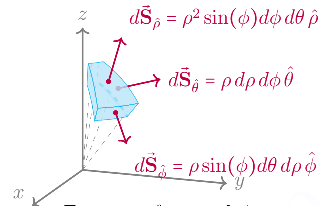
For $x = c$, $y = c$ and $z = c$.

Cylindrical:



For $r = c$, $\theta = c$ and $z = c$.

Spherical:



For $\rho = c$, $\theta = c$ and $\phi = c$.

Green's Theorem: Let $\partial\mathcal{D}$ be a simple, closed curve with counterclockwise orientation. Then

$$\oint_{\mathcal{C}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \oint_{\partial\mathcal{D}} P\,dx + Q\,dy = \iint_{\mathcal{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_{\mathcal{D}} \mathrm{Curl}(\vec{\mathbf{F}})_z\, dA$$

Stokes' Theorem: Let $\mathcal{S}$ be an oriented surface with smooth, simple closed boundary curves. Let $\vec{\mathbf{F}}$ be a vector field whose components have continuous partial derivatives. Then

$$\oint_{\partial\mathcal{S}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_{\mathcal{S}} \mathrm{Curl}(\vec{\mathbf{F}}) \cdot d\vec{\mathbf{S}}$$

where the components of $\partial\mathcal{S}$ are oriented using a right-hand-rule orientation.

The Divergence Theorem: Let $\mathcal{S}$ be a closed surface that encloses a solid $\mathcal{W}$ in $\mathbb{R}^3$. Assume that $\mathcal{S}$ is piecewise smooth and is oriented by normal vectors pointing outside $\mathcal{W}$. Let $\vec{\mathbf{F}}$ be a vector field whose domain contains $\mathcal{W}$. Then:

$$\oiint_{\mathcal{S}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iiint_{\mathcal{W}} \mathrm{Div}(\vec{\mathbf{F}})\, dV.$$

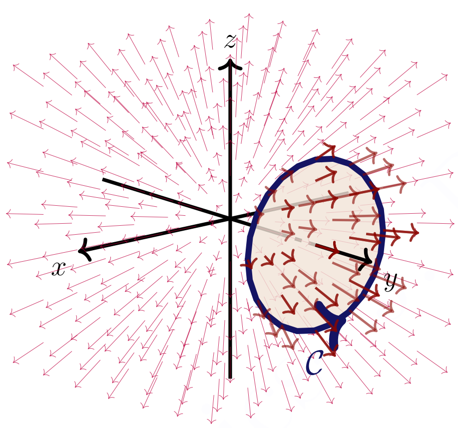
Group Work Portion of the Worksheet

Names: ____

Work in groups to do this portion of the worksheet. Make sure to take parts in solving the problems. Your participation score is a combination of being prepared, willing to explore the problem, working in groups and contributing toward the solution.

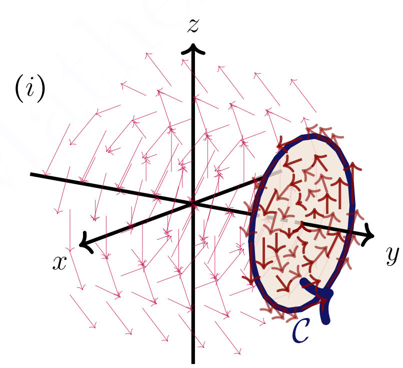
1. **Background Story:** Important Theorems for this problem: • Stokes' Theorem $\oint_{\partial \mathcal{S}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_{\mathcal{S}} \operatorname{curl}(\vec{\mathbf{F}}) \cdot d\vec{\mathbf{S}}$ and • Divergence Theorem $\oiint_{\mathcal{S}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iiint_{\mathcal{W}} \operatorname{div}(\vec{\mathbf{F}})\, dV$.

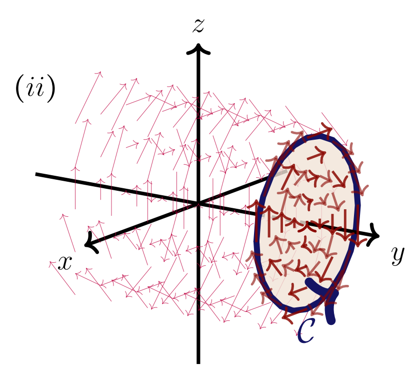
   (A) Remember $\vec{e}_r = \dfrac{\vec{\mathbf{r}}}{\|\vec{\mathbf{r}}\|}$ from previous worksheet! Use the information in those worksheets to quickly compute the contour line vector integral $\oint_{\mathcal{C}} \vec{e}_r \cdot d\vec{\mathbf{r}}$ where $\mathcal{C}$ is oriented as shown.



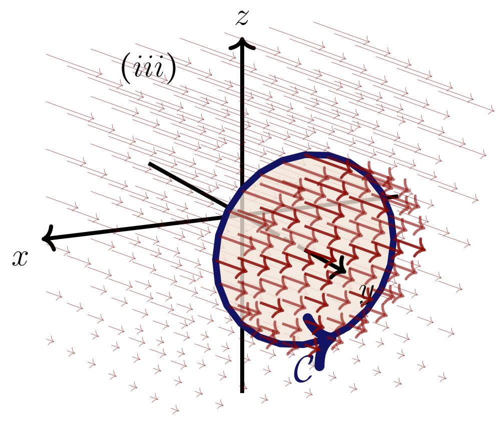
View the vector field in a 3-D app: https://www.geogebra.org/m/ffbkka56

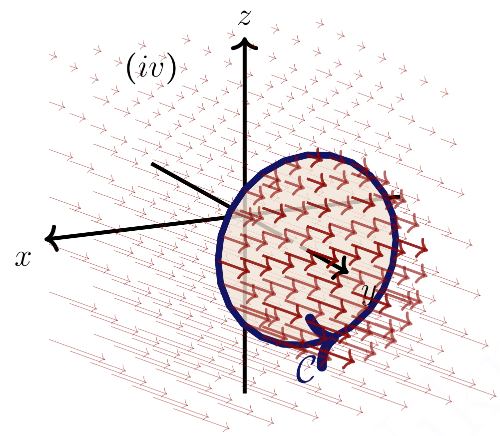
   (B) Determine in each case, whether $\oint_{\mathcal{C}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ is zero, positive or negative.[1]





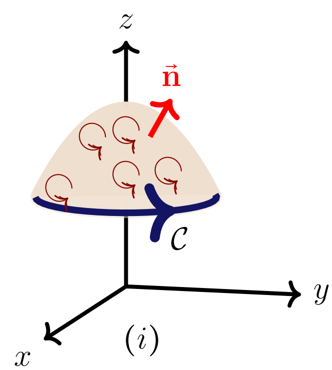
[1]The vector field is given here. Look at the vector field on the surface to estimate the direction of the curl. View the vector fields in 3-D App: https://www.geogebra.org/m/pnhpxm8g

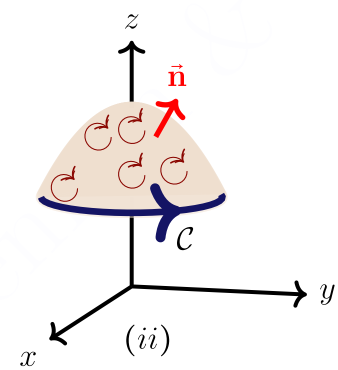




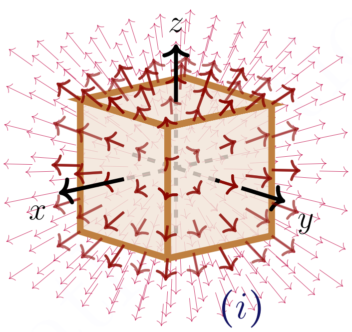
View the vector field in 3-D app: https://www.geogebra.org/m/ku6eezmg

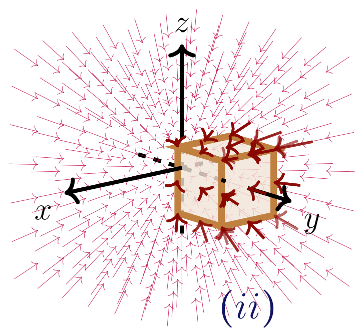
(C) Determine in each case, whether $\oint_{\mathcal{C}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ is zero, positive or negative.[2]





(D) Determine in each case, whether $\oiint_{\mathcal{S}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$, with outward orientation, is positive, negative and zero.





View the vector fields in 3-D app: https://www.geogebra.org/m/ecsrt4yt

[2]The infinitesimal rotations on the surface are given. Use them to estimate the direction of rotation on the boundary.

2. **Background Story:** One main purpose of this problem is to compute the surface integral (flux) with outward orientation on a closed surface and then compare the result to the result of the Divergence Theorem. Another purpose is to see computing $\vec{\mathbf{N}}$ for some of these surfaces that are important for physics is tedious and the memorization method can be handy (refer to memorization method in the above notes.).

   Questions: Find the vector surface integral $\displaystyle\iint_{\mathcal{S}} \langle y, x, 3z^2\rangle \cdot d\vec{\mathbf{S}}$ where $\mathcal{S}$ is parameterized by

   (A) $\vec{\mathbf{G}}(s,t) = \langle s\cos t, s\sin t, 1\rangle, \quad 0 \le s \le 1, \quad 0 \le t \le 2\pi$ oriented upward.

   Step 2: $\vec{\mathbf{G}}_s \times \vec{\mathbf{G}}_t = \langle \qquad , \qquad , \qquad \rangle$

   $\vec{\mathbf{N}} = \langle \qquad , \qquad , \qquad \rangle$

   Step 3: $\vec{\mathbf{F}}(\vec{\mathbf{G}}(s,t)) = \langle \qquad , \qquad , \qquad \rangle$

   Step 4: $\iint_{\mathcal{D}} \vec{\mathbf{F}}(\vec{\mathbf{G}}(s,t)) \cdot \vec{\mathbf{N}}\, dA_{st} =$

   (B) $\vec{\mathbf{H}}(s,t) = \langle \cos t, \sin t, s\rangle, \quad 0 \le s \le 1, \quad 0 \le t \le 2\pi$ oriented outward.

   Step 2: $\vec{\mathbf{H}}_s \times \vec{\mathbf{H}}_t = \langle \qquad , \qquad , \qquad \rangle$

   $\vec{\mathbf{N}} = \langle \qquad , \qquad , \qquad \rangle$

   Step 3: $\vec{\mathbf{F}}(\vec{\mathbf{H}}(s,t)) = \langle \qquad , \qquad , \qquad \rangle$

   Step 4: $\iint_{\mathcal{D}} \vec{\mathbf{F}}(\vec{\mathbf{H}}(s,t)) \cdot \vec{\mathbf{N}}\, dA_{st} =$

   (C) $\vec{\mathbf{I}}_3(s,t) = \langle s\cos t, s\sin t, 0\rangle, \quad 0 \le s \le 1, \quad 0 \le t \le 2\pi$ oriented downward.

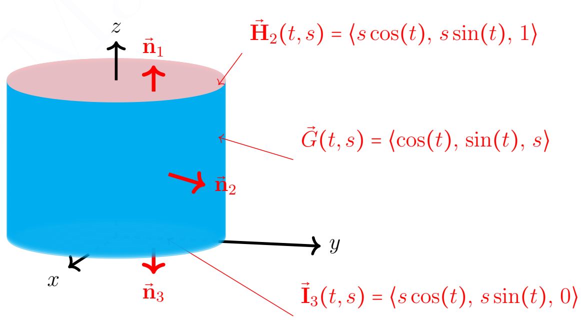
   Step 2: $\vec{\mathbf{I}}_s \times \vec{\mathbf{I}}_t = \langle \qquad , \qquad , \qquad \rangle$

   $\vec{\mathbf{N}} = \langle \qquad , \qquad , \qquad \rangle$

   Step 3: $\vec{\mathbf{F}}(\vec{\mathbf{I}}(s,t)) = \langle \qquad , \qquad , \qquad \rangle$

   Step 4: $\iint_{\mathcal{D}} \vec{\mathbf{F}}(\vec{\mathbf{I}}(s,t)) \cdot \vec{\mathbf{N}}\, dA_{st} =$

   (D) Verify that the sum of the three surface integral is equal to $\iiint_{\mathcal{S}} \operatorname{div}(\vec{\mathbf{F}})dV$, where $\mathcal{S}$ is the solid of the filled cylinder $x^2 + y^2 < 1$, $0 \le z \le 1$.[3]



[3] This is the verification of Divergence Theorem on the closed surface.

GroupWork Rubrics:

Preparedness: ——/0.5, Contribution: ——/0.5, Correct Answers: ——/0.5

Individual Portion of the Worksheet

Name: ________________

Upload this section individually on canvas or turn it in to your instructor on the 2nd lab day of the week. You can ask questions in class and work in groups but you turn in the individual work. Start before the class so you can ask questions during the class. If you didn't complete the work in class, make sure to work on it outside the class and complete it. Show all your work; your score depends on the work you have shown.

All Individual Questions are Posted in previous section.