Week 5-Lab 1 or 2: Worksheet 7: Section 14.5

They said: "What is the point of quizzes again?" I said: " To get you to study in small chunks in a low stake setting. (Each are 0.5% of your grade?!) "I also said: " Have you been doing the Diagnostic quizzes?"

Gradients and Directional Derivatives in 2 Variables

Let $f(x,y)$ be a scalar function of 2 variables. The **gradient vector** of differentiable function f at a point (a,b) in the domain of f is $\nabla f(a,b) = \langle f_x(a,b),\ f_y(a,b)\rangle$.

If $\vec{\mathbf{u}} = \langle p,q\rangle$ is a unit vector, then the **directional derivative** of f at (a,b) in the direction of $\vec{\mathbf{u}}$ is

$$D_{\vec{\mathbf{u}}} f(a,b) = \nabla f(a,b) \cdot \vec{\mathbf{u}}$$

Gradients and Directional Derivatives in 3 Variables

Let $f(x,y,z)$ be a function of 3 variables. The **gradient vector** of a scalar differentiable function f at a point (a,b,c) in the domain of f is

$$\nabla f(a,b,c) = \langle f_x(a,b,c),\ f_y(a,b,c),\ f_z(a,b,c)\rangle .$$

If $\vec{\mathbf{u}} = \langle p,q,r\rangle$ is a unit vector, then the **directional derivative** of f at (a,b,c) in the direction of $\vec{\mathbf{u}}$ is

$$D_{\vec{\mathbf{u}}} f(a,b,c) = \nabla f(a,b,c) \cdot \vec{\mathbf{u}}$$

Directions with Extreme Rates of Change

The gradient ∇f points in the direction that f is increasing fastest.

- The **largest** (**smallest**) directional derivative is in the direction $\nabla f(a,b)$ (or $-\nabla f(a,b)$) and equal to $\|\nabla f(a,b)\|$ (or $-\|\nabla f(a,b)\|$).

Tangent Planes and Normal Lines

We can find the tangent plane to any surface S defined by an equation in x,y,z (we do not need z to be a function of x and y).

Express the equation in the form $\underbrace{F(x,y,z) = \text{constant}}_{\text{A Level Surface}}$

Next, compute $\nabla F(x,y,z)$.

Then, the equation of the tangent plane at any point (a,b,c) on S is

$$\nabla F(a,b,c) \cdot \langle x-a,\ y-b,\ z-c\rangle = 0$$

and the normal line has equation

$$\vec{\mathbf{r}}(t) = \langle a,b,c\rangle + t\nabla F(a,b,c)$$

or equivalently

$$x = a + tF_x(a,b,c), \quad y = b + tF_y(a,b,c), \quad z = c + tF_z(a,b,c).$$

Group Work Portion of the Worksheet

Names: ____

Work in groups to do this portion of the worksheet. Make sure to take parts in solving the problems. Your participation score is a combination of being prepared, willing to explore the problem, working in groups and contributing toward the solution.

1. (A) **Background Story:** Find the tangent plane to an implicit surface which is not a graph of a function. Use gradient vector as the normal vector.

   Questions: Find the tangent plane to the surface of $x^2 + z^2 = y^2 + 9$ at $(2, -2, 3)$. [1]

   (B) **Background Story:** In Section 14.4 we gave the formula $z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$ and in Section 14.5, we introduced $\nabla g(a, b, f(a, b))$ as the normal vector to tangent plane at $(a, b, f(a, b))$ when $g(x, y, z) = z - f(x, y)$.[2]

   Questions: Compare the methods in Section 14.4 and 14.5 for finding the tangent plane to the graph of $z = f(x, y)$ at point $(a, b, f(a, b)$.

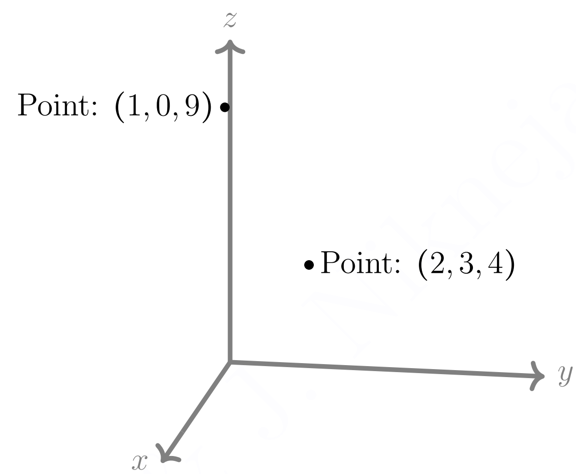
   (C) Which method is preferred if the surface is not a graph of a function.

[1] Video: https://youtu.be/cOFFPMRez6s
[2] Video: https://mediahub.ku.edu/media/t/1_aitpmjup

2. **Background Story:** The gradient of a scalar function is perpendicular to level surfaces of that scalar function.

   Questions: Consider the differentiable scalar function $R(x,y,z) = \sqrt{x^2+y^2+z^2}$.

   (A) Describe and sketch the level surfaces for R at $(2,3,4)$ and $(1,0,9)$.



   (B) Compute and sketch the gradient vectors $\nabla R(2,3,4)$ and $\nabla R(1,0,9)$.

   A Video:: https://youtu.be/NdeYuIi8TnQ

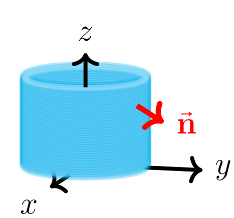
3. **Background Story:** This problem is shadowing the concept of cylindrical and spherical standard unit basis. **You need to know everyone of these by heart before we start Section 16.5.**

   Questions:

   In each of the following cases $\vec{\mathbf{n}}$ is a unit vector normal to the given surface. Compute $\vec{\mathbf{n}}$ then convert any $x, y,$ and z appearing in $\vec{\mathbf{n}}$ to **cylindrical** r, θ, z or **spherical** ρ, ϕ, θ. Use gradient vectors and the geometry of the shapes to find $\vec{\mathbf{n}}$ in each case. *(Simplify as much as possible; replace variables by constant when applicable.)*

   (A) $\vec{\mathbf{n}}$ is the outward **unit vector** normal to the surface of the cylinder $x^2+y^2=9$; find $\vec{\mathbf{n}}$ in terms of the point (x,y,z) on the surface; use **cylindrical** conversion formula to convert x,y,z of $\vec{\mathbf{n}}$ to r, θ, z.

   $\vec{\mathbf{n}}(x,y,z) = \langle$_____, _____. _____$\rangle$

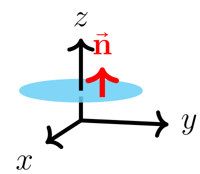


   $\vec{\mathbf{n}}(r,\theta,z) = \langle$_____, _____. _____$\rangle$[3]

[3] That is, $\vec{\mathbf{n}}(r,\theta,z) = \langle x(r,\theta,z), y(r,\theta,z), z(r,\theta,z)\rangle$.

(B) $\vec{\mathbf{n}}$ is the upward **unit vector** normal to the surface of the disk $x^2 + y^2 \leq 9$ and $z = 3$; find $\vec{\mathbf{n}}$ in terms of the point (x, y, z) on the surface; use **cylindrical** conversion formula convert to x, y, z of $\vec{\mathbf{n}}$ to r, θ, z.

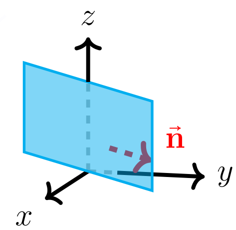
$\vec{\mathbf{n}}(x, y, z) = \langle$_____, _____. _____$\rangle$



$\vec{\mathbf{n}}(r, \theta, z) = \langle$_____, _____. _____$\rangle$

(C) $\vec{\mathbf{n}}$ is the **unit vector** normal to the surface of plane $y = \sqrt{3}x$, plane containing z-axis[4]; find $\vec{\mathbf{n}}$ in terms of the point (x, y, z) on the surface; use **cylindrical** conversion formula to convert x, y, z in $\vec{\mathbf{n}}$ to r, θ, z.

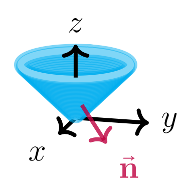
$\vec{\mathbf{n}}(x, y, z) = \langle$_____, _____. _____$\rangle$



$\vec{\mathbf{n}}(r, \theta, z) = \langle$_____, _____. _____$\rangle$

(D) $\vec{\mathbf{n}}$ is the outward **unit vector** in direction orthogonal to the surface cone $x^2 + y^2 = 3z^2$ [5]; find $\vec{\mathbf{n}}$ in terms of the point (x, y, z) on the surface, for $z > 0$; then convert x, y, z to **spherical** ρ, θ, ϕ.

$\vec{\mathbf{n}}(x, y, z) = \langle$_____, _____. _____$\rangle$



$\vec{\mathbf{n}}(\rho, \theta, \phi) = \langle$_____, _____. _____$\rangle$

A Video: https://mediahub.ku.edu/media/t/1_jblhnkzf

[4] That is the plane $\theta = \arctan(\sqrt{3}) = \dfrac{\pi}{3}$ or $\theta = \pi + \arctan(\sqrt{3}) = \pi + \dfrac{\pi}{3}$. Note that, in electromagnetism and other area of physics, we often compute unit normal vector to rotating rectangular regions or rotating disks. The vectors are going to be in terms of θ only; find them however you can.

[5] $\phi = \arctan(\sqrt{3}) = \dfrac{\pi}{3}$

4. The temperature at a point (x, y, z) is given by

$$T(x, y, z) = \frac{72900e^{\frac{-5y^2}{10}}}{x^2 + 9z^2}$$

where T is measured in $^\circ C$ and x, y, z in meters.

(A) Find the gradient vector of T at $P(3, 0, 3)$.

(B) Find the rate of change of temperature at the point P in the direction toward the point $(3, 0, -3)$.

(C) In which direction does the temperature increase fastest at P?

(D) Find the fastest rate of temperature increase or decrease at P.

Video: https://youtu.be/qJ4M9Iisa3w

5. **Background Story:** The gradient of a level curve is orthogonal to the level curve containing it. In this question, explore the concepts.

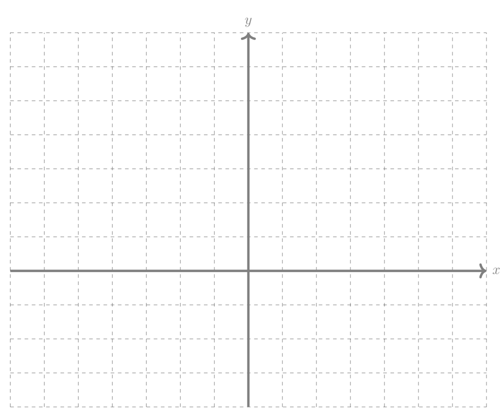
Questions: Consider $g(x, y) = x^2 + y - 4x$.

(A) Compute the gradient vector $\nabla g(2, 5)$.

(B) Is $(2, 5)$ on the level curve $g(x, y) = 1$?

(C) Use the $\nabla g(2, 5)$ to find the tangent line to the curve $g(x, y) = 1$ at $(2, 5)$.

(D) Sketch the level curve $g(x, y) = 1$ and the $\nabla g(2, 5)$ at point $(2, 5)$.



GroupWork Rubrics:

Preparedness: ___/0.5, Contribution: ___/0.5, Correct Answers: ___/0.5

Individual Portion of the Worksheet

Name: ________________

Upload this section individually on canvas or turn it in to your instructor on the 2nd lab day of the week. You can ask questions in class and work in groups but you turn in the individual work. Start before the class so you can ask questions during the class. If you didn't complete the work in class, make sure to work on it outside the class and complete it. Show all your work; your score depends on the work you have shown.

GroupWork Rubrics day 2:

Preparedness: ___/0.5, Contribution: ___/0.5, Correct Answers: ___/0.5

6. **Background Story:** The gradient of a level curve is orthogonal to the level curve containing it. In this question, explore the concepts.

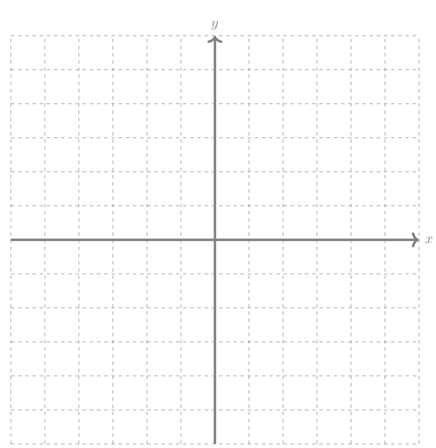
   Questions: Consider $g(x,y) = x^2 + y^2 - 5x$.

   (A) *(1.5 points) Find the gradient vector of g at $P(3,2)$.*

   (B) (0.5 points) Is $(3,2)$ on the level curve $g(x,y) = -2$? Why?

   (C) (0.5 points) Use the $\nabla g(3,2)$ to find the tangent line to the curve $g(x,y) = -2$ at $(3,2)$.

   (D) (1 point) Sketch the level curve $g(x,y) = -2$ and the $\nabla g(3,2)$ at point $(3,2)$.



7. The temperature at a point (x, y, z) in $\mathbf{R}^3$ near a heat source is given by

$$T(x, y, z) = \frac{120}{\sqrt{13 + x^2 + y^2 + z^2}}$$

Where T is measured in $°C$ and x, y in meters.

(A) *(1 point)* Find the gradient vector at an arbitrary point (x, y, z) $(\nabla T(x, y, z))$.

(B) *(1 point)* Find the equation of the level surface where temperature is $15°C$; simplify and describe the surface.

(C) *(0.25 points)* The heat source is a single point in 3D space; what point is that?

(D) *(0.75 points)* Let $\vec{\mathbf{n}}$ be the unit vector normal to the surface $T(x, y, z) = 15$ at a point (x, y, z) on the surface in direction away from the heat source. Give $\vec{\mathbf{n}}$ in terms of x, y, and z. *Simplify as much as possible.*

(E) *(0.5 points)* What is the direction of fastest increase in temperature at point $(3, 1, 4)$.