Week 6-Lab 2: Worksheet 9: Section 14.7

They said: "I feel I learned some concept; but next time I try it, I will forget it." I said: "Keep going to lectures, going to labs, doing assignments, using the help room, and talking about math with your friends and us. Hear and do the same concept over and over; every time you learn it a bit deeper until it really clicks. I have three words for you; repetition, repetition, and repetition."

Local and Absolute Extrema

Let $f(x,y)$ be a function of two variables, with domain D.
A point (a,b) in D is...

- a **local maximum** if $f(x,y) \leq f(a,b)$ for (x,y) near (a,b);
- a **local minimum** if $f(x,y) \geq f(a,b)$ for (x,y) near (a,b);
- an **absolute maximum** if $f(x,y) \leq f(a,b)$ for all (x,y) in D;
- an **absolute minimum** if $f(x,y) \geq f(a,b)$ for all (x,y) in D.

The Second Derivative Test

Let $f(x,y)$ be a function of two variables. The **discriminant** of f at a point (a,b) in the domain is

$$D(a,b) = \begin{vmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{vmatrix} = f_{xx}(a,b)f_{yy}(a,b) - [f_{xy}(a,b)]^2 .$$

Second Derivative Test If (a,b) is a critical point of f and all second partials f_{xx}, f_{xy}, f_{yy} are continuous near (a,b), then

(I) If $D(a,b) > 0$ and $f_{xx}(a,b) > 0$, then (a,b) is a **local minimum.**

(II) If $D(a,b) > 0$ and $f_{xx}(a,b) < 0$, then (a,b) is a **local maximum.**

(III) If $D(a,b) < 0$, then (a,b) is a **saddle point.**

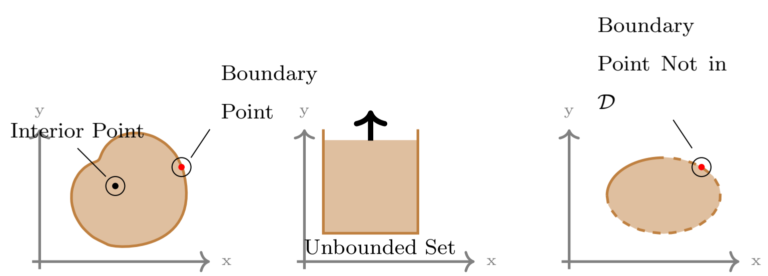
(IV) If $D(a,b) = 0$, then the test is **inconclusive.**

Sketch of the proof: https://youtu.be/1jC4_NaFpks

Extreme Value Theorem

If $z = f(x,y)$ is continuous on a **closed** and **bounded** set D in $\mathbb{R}^2$, then $f(x,y)$ attains an absolute maximum and an absolute minimum.

- "Closed" means that D contains all the points on its boundary.
- "Bounded" means that D does not go off to infinity in some direction.
  (Disks are bounded; so is any set contained in *some* disk.)



The Closed/Bounded Domain Method

Extreme Value Theorem

If $z = f(x, y)$ is continuous on a **closed** and **bounded** set D in $\mathbb{R}^2$, then $f(x, y)$ attains an absolute maximum and an absolute minimum.

Closed/Bounded Domain Method to find absolute extrema:

(I) Find all critical points.

(II) Find the extrema of f on the **boundary** of D.

(III) The points found from (I) and (II) with the largest/smallest value(s) of f are the absolute extrema.

The Second Derivative test isn't required.

Summary video: https://youtu.be/gM9Jc2RGi2c

Group Work Portion of the Worksheet

Names: ____

Work in groups to do this portion of the worksheet. Make sure to take parts in solving the problems. Your participation score is a combination of being prepared, willing to explore the problem, working in groups and contributing toward the solution.

1. **Background Story:** Follow the steps of computing the local extrema using the second derivative test: ① Compute the gradient. ② Compute the critical points by setting each term of gradient equal to zero and solving. ③ Compute the second derivatives and form the discriminant. ④ Classify each critical point using the discriminant.

   Questions:

   Find any local maxima, local minima, and saddle points for

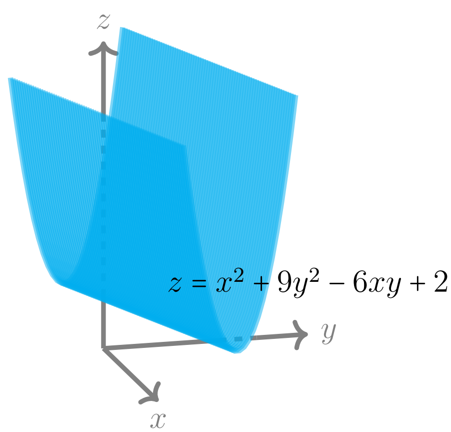
   $$f(x, y) = x^3 + y^2 + 2xy - 6x - 5y + 7.$$

   Video: https://youtu.be/A-xzmu4PkhE

2. **Background Story:** We study 3 different methods of finding extrema in Sections 14.7 and 14.8. This included the second derivative test for local extrema which can give an inconclusive result. In Calculus of multivariable (Calculus III), unlike Calculus of single variable (Calculus I), a first derivative does not exists and instead we may be able to include a study of shapes. The next problem is a situation when the discriminant is zero and second derivative test is inconclusive.

   Questions: This problem concerns the extrema of $f(x,y) = x^2 + 9y^2 - 6xy + 2$.

   (a) Discuss the problem and mark all the minimum points on the graph of the function.



   (b) Algebraically show that $f(x,y) = x^2 + 9y^2 - 6xy + 2$ has an infinite number of critical points and that $D = 0$ at each one.

   (c) By completing a square, show that f has an absolute minimum value at each critical point. *(Hint: Start with completing a square.)*

3. **Background Story:** In the following problem we can use the method of closed and bounded region to solve for absolute extrema. The boundary of the region consists of many pieces and each piece has to be parameterize separately.

   We suggest that you use a table to keep track of parameterization of each piece of boundary. We suggest to use the domain diagram and your knowledge of the shape of the graph to see if your answers make sense.

   Questions: Find the absolute maxima and absolute minima of $f(x,y) = 4x + 6y - x^2 - y^2$ on the region bounded by the line $x = 0$, $x = 6$, $y = 0$, and $y = 5$ by following the steps.

   (A) Explain why the conditions for method of closed and bounded regions is satisfied.

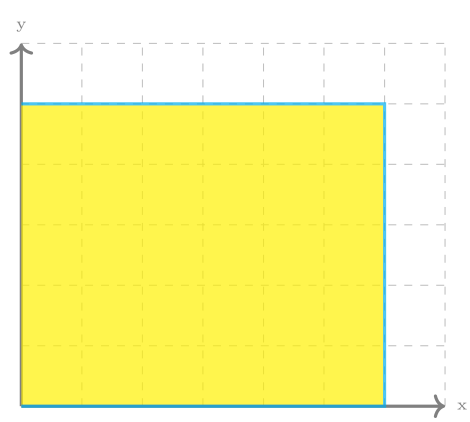
   (B) Find the critical points of f. Is the critical point in the domain?

   (C) Parameterize each piece of boundary and find the critical points for each piece.

   (D) All corners of the boundary are counted as critical points of the boundary. What are those points?

   (E) Evaluate f in all critical and end points and classify the absolute extrema.

   (F) What is the shape of the graph of f? (Hint: The graph is a quadric surfaces.) Does the information that you found in Part (E) match the information about the graph and domain.



   Video: https://youtu.be/HmLfLEkOTU4

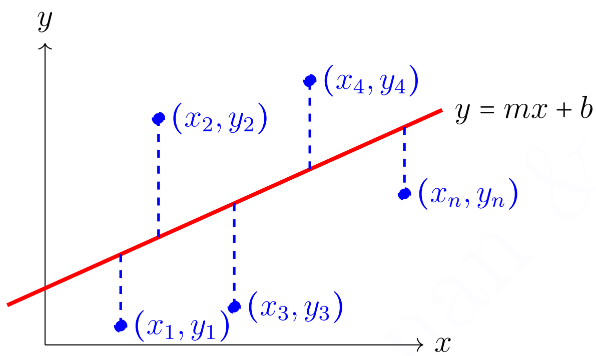
4. **Background Story:** One of the application of optimization is in minimizing the error in prediction (data science). The following example helps deriving the formula for linear regression for one independent variable.

Questions:

Given n data points $(x_1, y_1), \cdots (x_n, y_n)$, the **linear least-squares fit** is the linear function

$$f(x) = mx + b \qquad \text{that minimizes} \qquad E(m,b) = \sum_{j=1}^{n} (y_j - f(x_j))^2$$

($E(m,b)$ is the the sum of the squares of lengths of the blue line segments in the picture.)



(A) Find critical point(s), (m,b), of the function of two variables, $E(m,b) = \sum_{j=1}^{n} (y_j - mx_j - b)^2$, by giving their equations of $E_m(m,b) = 0$ and $E_b(m,b) = 0$.

(B) How do you solve for (m,b) in Part (A)? What kind of system of equations did you find in Part (A)?

Use these formulas to solve a problem like this on your Achieve homework.

GroupWork Rubrics:

Preparedness: ——/0.5, Contribution: ——/0.5, Correct Answers: ——/0.5

Individual Portion of the Worksheet

Name: ________________

Upload this section individually on canvas or turn it in to your instructor on the 2nd lab day of the week. You can ask questions in class and work in groups but you turn in the individual work. Start before the class so you can ask questions during the class. If you didn't complete the work in class, make sure to work on it outside the class and complete it. Show all your work; your score depends on the work you have shown.

5. *(3.5 points) Find the absolute maximum and absolute minimum values of* $f(x, y) = 3xy - 3x - 3y + 1$ *on the region inside the triangle with vertices* $(0, 0)$, $(5, 0)$ *and* $(0, 4)$.