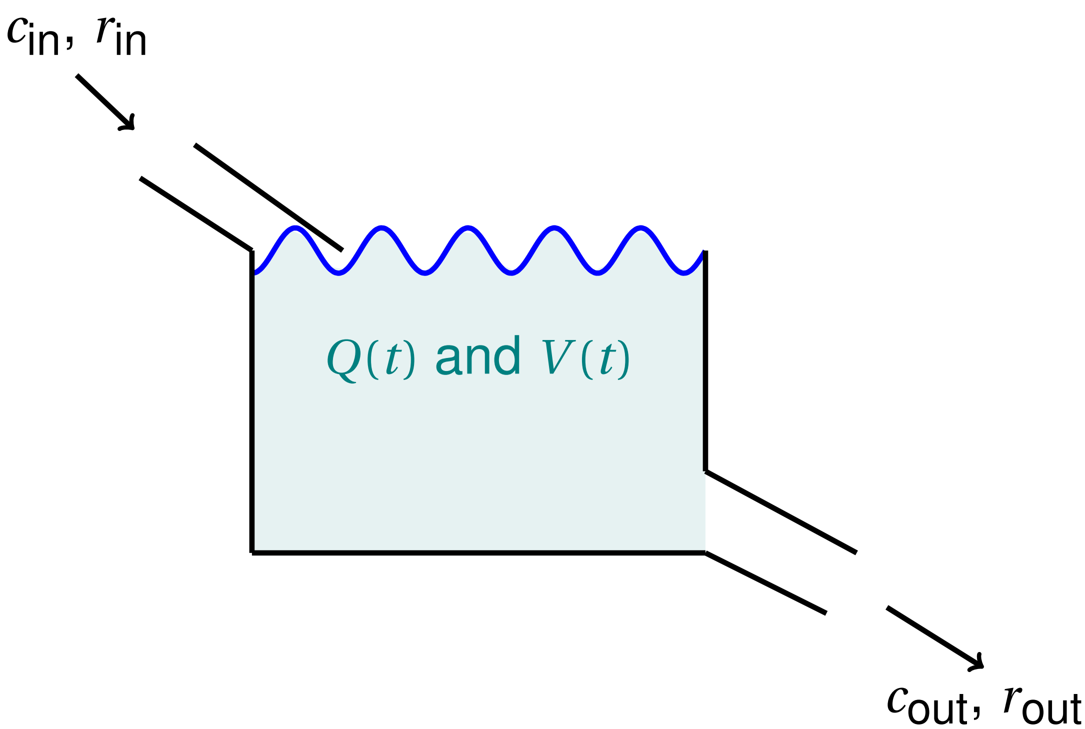
c_{in}, r_{in}
$Q(t)$ and $V(t)$
c_{out}, r_{out}

Example 1

Consider a tank with volume $100L$ is filled with a salt solution. Suppose a solution with $2g/L$ of salt flows into the tank at a rate of $5L/min$. The solution in the tank is well-mixed. Solution flows out of the tank at a rate of $5L/min$. If initially there is $10g$ of salt in the tank, how much salt will be in the tank as a function of time?

What is the limiting amount of salt in the tank?

At what time amount of salt in the tank is $100g$?

Solution

- Let $Q(t)$ = The amount of salt inside the tank at time t.

- Amount of salt coming in $= 5 \times 2 = 10$ gr/min

- Amount of salt leaving $= 5 \times \dfrac{Q(t)}{100} = 0.05Q(t)$ gr/min

- change in amount per minute = amount **coming in** per minute – amount **leaving** per minute

- $\dfrac{dQ(t)}{dt} = 10 - 0.05Q(t)$

- You can use the method that we used for the falling objects but I recommend Linear First Order.

Step 1 : Standard form: $Q' + 0.05Q = 10$

Step 2 : $\mu(t) = e^{\int 0.05dt} = e^{0.05t}$

Step 3 : Multiply: $Qe^{0.05t} + 0.05e^{0.05t}Q = 10e^{0.05t}$

Step 4 : Check: $(Qe^{0.05t})' =$LHS

Step 5 : Replace LHS: $(Qe^{0.05t})' = 10e^{0.05t}$ and integrate
$Qe^{0.05t} = 200e^{0.05t} + C$

Step 6 :Solve explicitly: $Q = 200 + Ce^{-0.05t}$

Step 7 : Use the initial value $Q(0) = 10$ to get $10 = 200 + C$.
That is, $\boxed{Q(t) = 200 - 190e^{-0.05t}}$ is the amount of salt in the tank at time t.

- The **limiting amount** can be found two different ways:

  1. The **limiting amount** is when the concentration of salt in the tank is equal to to the incoming concentration: $\lim_{t\to\infty} Q = 2 \times 100 = \boxed{200}$ gr

  2. Or $\lim_{t\to\infty} Q(t) = 200 - 190 \lim_{t\to\infty} e^{-0.05t} = \boxed{200}$ gr

  Let T be the **time** when $Q(T) = 100$ and solve $100 = 200 - 190e^{-0.05T}$ to get $T = -\frac{\ln(100/190)}{0.05} \simeq 12.8371$ minutes

Example 2

A tank of Capacity of 500 gal originally contains 200 gal of water with 100 oz of salt in solution. Water containing 1 oz/gal enters at rate of 3 gal/min. well mixed solution leaves at the rate 2 gal/min. Find the concentration at any time before the overflow.

After 200 min the process is stopped, and the fresh water is poured into the tank at the rate of 2 gal/min, with the mixture leaving at the same rate. Find the amount of salt in the tank at the end of an additional 200 min.

Solution:

- Let Q be the amount of salt in the tank. The volume of the liquid in the tank at time t is $V(t) = 200+t$. The amount of salt coming in per minute is $3gal/min \times 1oz/gal = 3oz/min$. The amount of salt going out per minute is $2 \times \frac{Q}{V} = \frac{2Q}{200+t} oz/min$.

- The ode is $\frac{dQ}{dt} = 3 - 2\frac{Q}{200+t}$ which is a linear ode; solve:

Step 1 : **standard form** $\frac{dQ}{dt} + 2Q(\frac{1}{200+t}) = 3$

Step 2 : **Integrating factor**

$\mu(t) = e^{2\ln(200+t)} = (200+t)^2$

Step 3 : **Multiply**: $(200+t)^2 Q' + 2(200+t)Q = 3(200+t)^2$

Step 4 : **Check** the left hand side: $(y(200+t)^2)' \stackrel{?}{=}$ LHS ✓

Step 5 : **Replace** LHS and **integrate** : $(Qe^{2\ln(200+t)})' = 3(200+t)^2$ gives $Q(200+t)^2 = (200+t)^3 + C$

Step 6 : **Explicit** solution: $Q = (200+t) + c(200+t)^{-2}$

Step 7 : **IVP** $Q(0) = 200 + c/200^2$

$Q(0) = 100$ so $c = -100(200^2)$

So the IVP solution is $\boxed{Q(t) = (200+t) - \frac{4\times 10^6}{(200+t)^2}}$

(Amount of salt at time t.)

If the tank was infinitely big, here we see that amount would approach infinity as $t \to \infty$ but since the tank is not infinitely big, that does not happen.

Concentration at any time: $\boxed{C(t) = 1 - \frac{4\times 10^6}{(200+t)^3}}$ Here we see that the concentration approaches 1 as $t \to \infty$. But that is not possible neither since the tank is not endless.

Note that after 200 min, there will be $Q(200) = 400 - 25 = 375$ oz of salt in the tank.

Solution, Part two, time $t+200$

- Let Q be the amount of salt in the tank. The volume of the liquid in the tank at time $t+200$ is $V = 400$. The amount of salt coming in per min=0. The amount of salt going out per minute is $2 \times \frac{Q}{V} = \frac{2Q}{200+200} oz/min$.

- The ode is $\frac{dQ}{dt} = 0 - 2\frac{Q}{400}$ which is a linear ode; solve:

Step 1 : **standard form:** $\frac{dQ}{dt} + Q(\frac{1}{200}) = 0$

Step 2 : **Integrating factor:** $\mu(t) = e^{0.005t}$

Step 3 : **Multiply:** $e^{0.005t}Q' + 0.005e^{0.005t}Q = 0$

Step 4 : **Check the left hand side:** $(ye^{0.005t})' \stackrel{?}{=}$ LHS $\checkmark$

Step 5 : **Replace** LHS and **integrate:** $(Qe^{0.005t})' = 0$ gives $Qe^{0.005t} = c$

Step 6 : **Explicit** solution: $Q = ce^{-0.005t}$

Step 7 : **IVP** To find the initial value $Q(0) = c$

$Q(0) = 375$ so $c = 375$

So the IVP is $\boxed{Q = 375e^{-0.005t}}$ which is the amount of salt at time t.

If the tank was infinitely big, here we see that amount would approach infinity as $t \to \infty$ but since the tank is not infinitely big, that does not happen.

The amount of salt in the tank after additional 200 minutes, $\boxed{Q(200) = 375e^{-0.005 \times 200} = 375e^{-1} \simeq 138}$ oz