Bernoulli equations: Steps

Recognition : An equation of the form $\boxed{y' + p(t)y = g(t)y^n}$ Where $n \neq 0$ and $n \neq 1$.*

Substitution : Use v-substitute: $\boxed{v = y^{1-n}}$ and use chain rule to take the derivative $\boxed{v' = (1-n)y^{-n}y'}$

Next divide every term of the original equation by y^n to get:

$y'y^{-n} + p(t)y^{1-n} = g(t)$ Now replace $y'y^{-n}$ by $\frac{v'}{1-n}$ and y^{1-n} by v to get: $\dfrac{v'}{1-n} + p(t)v = g(t)$

Linear Eq: Multiply by $1-n$ to get $\boxed{v' + (1-n)p(t)v = (1-n)g(t)}$ which is a linear equation. Solve for v.

Back-sub: Replace v by y^{1-n}. Then solve for y.

*$n = 0$ and $n = 1$ means that the equation a linear equation.

- Solve $y' = 5y - 5xy^3, \;\; y(0) = 1$.

Solution:

Recognition: $y' - 5y = -5xy^3$ This is a Bernoulli equation with $n = 3$, $p(x) = -5$, $q(x) = -5x$.

Choose Sub. : We make the substitution.

Divide both sides by the highest power of y.

$\frac{y'}{y^3} - \frac{5}{y^2} = -5x$

$v = y^{-2}$ (You can either use formula $1 - n = 1 - 3$ or the power of y in the second term f the equation.)

Applying the chain rule, we have

$v' = -2y^{-3}y'$

Solving for y', we have

$y' = -\frac{1}{2}y^3 v'$

Substitute : Substituting for y' in the differential equation we have

$-\frac{1}{2}v' - 5y^{-2} = -5x$

Dividing both sides by $-.5$, we have

$v' + 10y^{-2} - 10x$

Note that $\frac{1}{y^2} = v$. Hence, we have

$v' + 10v = 10x$

Linear : Solve the linear ode for the dependent variable v. The solution is (I am omitting this part since you are proficient now.)

$$v' + 10v = 10x \implies v = Ce^{-10x} + x - \frac{1}{10}$$

Back-sub. : Substituting $v = 1/y^2$, we have

$$\boxed{y^{-2} = Ce^{-10x} + x - \frac{1}{10}}$$ (Implicit General Sol.)

$$\boxed{y = \pm\sqrt{\frac{1}{Ce^{-10x}+x-\frac{1}{10}}}}$$ (Explicit General Sol.)

IVP : $1 = \pm\sqrt{\frac{1}{Ce^{-10(0)}+(0)-\frac{1}{10}}} \implies C = \frac{11}{10}$ and $+$ sign.

So $$\boxed{y = \sqrt{\frac{1}{\frac{11}{10}e^{-10x}+x-\frac{1}{10}}}}$$ (IVP solution)

- Solve with Bernoulli method $y' + 2xy + xy^4 = 0$.

  Solution:

Bernoulli Form : $y' + 2xy = -xy^4$

Divide by y^4 : $\frac{y'}{y^4} + 2\frac{x}{y^3} = -x$.

Substitution : $v = y^{-3} \implies v' = -3y^{-4}y'$

$$\frac{-v'}{3} + 2xv = -x$$

Linear Form : $v' - 6xv = 3x$

$\mu(x)$: $\mu(x) = e^{-3x^2}$.

$e^{-3x^2}y = \int 3xe^{-3x^2}\,dx$ (Use u-sub $u = -3x^2$.)

$$\boxed{y^{-3} = -1/2 + Ce^{3x^2}}$$

Also note that this equation was also Separable. I would like to note that we generally choose separable method over Bernoulli method* but in this case integral associated with separable method is somewhat difficult.

$$-\frac{dy}{x^4+2x} = xdx$$

Integrating the left hand side is not as easy and requires a fairly complicated partial fraction. Try using wolfram to see that.

*I also liked this to be solved as a Bernoulli equation because of its simplicity as a Bernoulli equation.