Method of Solving Exact Equations

Form : The Standard form is $M(x,y)dx + N(x,y)dy = 0$.

Recognition : If $M_y = N_x$ and M and N have continuous partial derivatives.

To check if the equation is exact, first subtract the right hand side from both side of the equation to get an equation equal to zero as in above. Sometimes you need to multiply by the denominator.

Solution : The goal is to find a function $\boxed{\psi(x,y) = c}$ such that the differential of $\psi(x,y)$ is the left hand side of the above equation. That is $d\psi(x,y) = \frac{\partial\psi(x,y)}{\partial x}dx + \frac{\partial\psi(x,y)}{\partial y}dy = 0$ where $\frac{\partial\psi(x,y)}{\partial x} = M$ and $\frac{\partial\psi(x,y)}{\partial y} = N$. This is also called the potential function.

To find the solution:

Step 1: Write in standard form and Check if $M_y \stackrel{?}{=} N_x$. If this is **not true**, the equation is **not** exact.

Step 2: Integrate M: $\boxed{\psi(x,y) = \int M(x,y)\,dx + h(y)}$ where $h(y)$ is a function of y only.

Step 3: Then take the partial with respect to y of ψ found in Step 2:

$$\boxed{\frac{\partial \psi(x,y)}{\partial y} = \frac{\partial\left(\int M(x,y)\,dx\right)}{\partial y} + h'(y)}$$

Step 4: We are looking for $\psi(x,y)$ such that $\frac{\partial \psi(x,y)}{\partial y} = N(x,y)$ so set the derivative found in Step 3 equal to N.

So set $N(x,y) = \dfrac{\partial\left(\int M(x,y)\,dx\right)}{\partial y} + h'(y)$

Step 5: Solve for $h'(y)$ in the equation in step 4. Note that $h'(y)$ is a function of y only. Then Integrate to find $h(y) = \int h'(y)dy$

Step 6: When you found $h(y)$ in Step 5, replace h in the formula for $\psi(x,y) = \int M(x,y)dx + h(y)$

Step 7: Set ψ you found in Step 6 equal to C.

$$\boxed{\int M(x,y)dx + h(y) = c}$$

Examples:

- $(y\cos(x) + 2xe^y)dx + (\sin x + x^2e^y + 2)dy = 0$

Solution:

Step 1: First, we check if the formula is exact. We have

$$M(x,y) = (y\cos(x) + 2xe^y) \quad \Rightarrow \quad M_y = \cos x + 2xe^y$$

$$N(x,y) = (\sin x + x^2e^y + 2) \quad \Rightarrow \quad N_x = \cos x + 2xe^y$$

Since the two partial derivatives are equal (and continuous) the equation is exact.

Step 2: Integrating M with respect to x, we have

$$\psi(x,y) = \int (y\cos(x) + 2xe^y)dx = y\sin(x) + x^2e^y + h(y)$$

Step 3: Take the derivative of $\psi(x,y)$ with respect to y:

$$\psi_y = \sin(x) + x^2e^y + h'(y)$$

Step 4: Set $\psi_y = N(x,y)$:

$$\sin(x) + x^2e^y + h'(y) = \sin x + x^2e^y + 2$$

Step 5: So $h'(y) = 2$. That is, $h(y) = 2y + c$

Step 6: The final solution to the ode is

$$\psi(x, y) = y\sin(x) + x^2 e^y + 2y$$

Step 7: $\boxed{y\sin(x) + x^2 e^y + 2y = C}$

This is an implicit solution. The closed form of explicit solution is not possible.

Other Examples:

- $(2xy^2+2y)+(2x^2y+2x)y'=0$ (For you to practice.)

  Convert to $(2xy^2+2y)dx+(2x^2y+2x)dy=0$ first.

- $(e^x\ln(y)+\frac{1}{x})dx+(\dfrac{e^x}{y}+2y)dy=0,\ y(1)=2$ and $x>0$. (I will do this one in class only.)

  The IVP solution is $\boxed{e^x\ln(y)+\ln(x)+y^2=e\ln(2)+4}$

Examples with Parametrs:

- $\frac{dy}{dx} = -\frac{ax-by}{bx-cy}, \ x > 0$. (**Not exact**. The method to solve this is called Homogeneous first order.)

- $\frac{dy}{dx} = \frac{ax-by}{bx-cy}$ (Exact. Left as homework assignment.)

The last two example demonstrate that with very little change, the nature of the method has to change.