Integrating factor for Exact ODEs

Why do we use integration factor? For similar reasons to linear equations. We find the expression which simplifies to make the ode exact. Those expressions are not very easy to find.

That is, if $\mu(x,y)$ is an integrating factor for $P(x,y)dx+Q(x,y)dx=0$, then $\mu(x,y)P(x,y)+\mu(x,y)Q(x,y)=0$ is exact.

There are some integrating factors such as the one in linear method and separable equations that have a very clear criteria methods. The rest are mostly try and error methods.

If there exists expressions that are a function of x or y, there are some known ways to find them.*

We discuss a function of y only. Other type of functions follow somewhat similarly.

*There some methods with certain substitutions and symmetries as well that we are not discussing.

Consider $P(x,y)\ dx+Q(x,y)\ dy=0$, our goal is to find an integrating factor $\mu(y)$, if it exists, such that $\mu(y)Q(x,y)dy+\mu(y)P(x,y)dx = 0$ becomes an **exact** ode. That is, $(\mu N)_x=(\mu M)_y$.

Now $N_x=(\mu(y)Q(x,y))_x=\mu Q_x$ and $M_y=(\mu(y)P(x,y))_y=\mu_y P+\mu P_y$ so

$\mu Q_x=\mu_y P+\mu P_y$.

Solving for μ_y gives:

$\mu_y=\mu\dfrac{Q_x-P_y}{P}$. Now if $\dfrac{Q_x-P_y}{P}$ is a function of y only, then we can treat this as a separable equation with y as independent variable and μ as dependent variable.

That is, $\dfrac{\mu_y}{\mu}=\dfrac{Q_x-P_y}{P} \implies \displaystyle\int\frac{\mu_y}{\mu}dy=\int\frac{Q_x-P_y}{P}dy$

$\implies \ln|\mu|=\int\frac{N_x-M_y}{d}y \implies |\mu|=e^{\displaystyle\int\frac{N_x-M_y}{M}dy}$.

(The absolute value can be removed because a negative sign does not change the outcome.)

So if $\boxed{\dfrac{Q_x - P_y}{P}}$ is a function of $\mathbf{y}$ **only**, then $\boxed{\mu = e^{\int \frac{Q_x - P_y}{P} dy}}$ is an integration factor and

$\mu(y)Q(x, y)dy + \mu(y)P(x, y)dx = 0$ will be solved using exact method.

Finding the integrating factor if it is a function of x only, follows similarly. In that case, the formula for integration factor is $\mu = e^{\int \frac{P_y - Q_x}{Q} dx}$.

So if $\boxed{\dfrac{P_y - Q_x}{Q}}$ is a function of $\mathbf{x}$ **only**, then $\boxed{\mu = e^{\int \frac{P_y - Q_x}{Q} dx}}$ is an integration factor and

$\mu(x)Q(x, y)dy + \mu(x)P(x, y)dx = 0$ will be solved using exact method.

Example

Find an integrating factor and solve the given equation.*

$(y+2)\sin(x)\,dy + y\cos(x)\,dx = 0$

Solution:

First check if an integrating factor as a function of x exists.

$P = y\cos(x) \qquad P_y = \cos(x)$

$Q = (y+2)\sin(x) \qquad Q_x = (y+2)\cos(x)$

$\mu(x) = e^{\int \frac{P_y - Q_x}{Q}dx} = e^{\int \frac{\cos(x)(-y-1)}{(y+2)\sin(x)}dx}$ A contradiction. Since $\mu(x)$ was assumed to be a function of x only.

*This equation is a separable equation and one of its obvious integrating factor is $\mu(x,y) = \dfrac{1}{y\sin(x)}$ to separate the variable. We find another integrating factor for the equation. The point of this example is to have an easy example for exploring the concepts that we just learned.

Now try to find $\mu(y)$ as a function of y:

$\mu(y) = e^{\int \frac{Q_x - P_y}{P} dy} = e^{\int \frac{\cos(x)(y+1)}{y\cos(x)} dy} = e^{y+\ln|y|} = |y|e^y$. Again absolute value and integrating factor. $\mu(y) = ye^y$ (Don't need them.)

Now multiply the integrating factor: $ye^y(y+2)\sin(x)dy + y^2e^y\cos(x)dx = 0$

New $P = y^2e^y\cos(x)$ $\qquad$ $P_y = 2ye^y\cos(x) + y^2e^y\cos(x)$

New $Q = ye^y(y+2))\sin(x)$ $\qquad$ $Q_x = y(y+2)e^y\cos(x)$

EXACT.

Find $\psi = y^2e^y\sin(x) + h(y)$

$y(y+2)e^y\sin(x) + h'(y) = y(y+2)e^y$

$h' = 0$ implies the solution is $y^2e^y\sin(x) = C$ and $y = 0$ is an equilibrium solution. $y \neq -2$ is a singular solution.

Example $y+(2xy-e^{-2y})y'=0$ *

$P=y$ and $Q=(2xy-e^{-2y})$

Solution:

$\mu=e^{\int \frac{P_y-Q_x}{Q}\,dx}=e^{\int \frac{2y-1}{2xy-e^{-2y}}\,dx}$ is not a function of x only

$\mu=e^{\int \frac{Q_x-P_y}{P}\,dy}=e^{\int \frac{2y-1}{y}}=\frac{e^{2y}}{y}$ is a function of y only.

Multiply by the second one: $\frac{e^{2y}}{y}y+\frac{e^{2y}}{y}(2xy-e^{-2y})y'=0$

to get : $e^{2y}dx+(2xe^{2y}-1/y)dy=0$

Solve the exact equation to get: $\psi=\boxed{xe^{2y}-\ln|y|=C}$ and $y=0$.

*In this one if you switch x to dependent and y to independent variable, the equation becomes linear with respect to x. we will show how the integrating factor is obtained without the change of role for variables.