First order homogeneous equations.

An ode of order one is homogeneous if you can write it in the following form:

$$\boxed{y' = f\left(\frac{y}{x}\right)}$$

- **How to recognize:** Solve the ode for y'.

  Most common form* is a quotient function f: $y' = \dfrac{P(x,y)}{Q(x,y)}$ where the total exponent of every term of the numerator ($P(x,y)$) and denominator ($Q(x,y)$) is the same then, the ode is homogeneous.

  To write the equation in the form above you may have to divide both numerator and the denominator by x to the power total exponent of any of the terms.

*You will have a homework assignment that is not of this form. You have to figure out the similarities yourself.

- v substitute: $\boxed{v = \frac{y}{x}}$

  Note that to find v' : First write $y = xv$ then use the product rule to take the derivative:

  $\boxed{y' = xv' + v}$

  After substitution $xv' + v = f(v)$ which is a separable equation which should be solved for v:

- Subtract v from both sides to get $xv' = f(v) - v$

- Put the RHS=f(v)-v in one fraction by using the common denominator.

- Divide both sides by x and the new fraction for $f(v) - v$.

- $\frac{dv}{f(v) - v} = \frac{dx}{x}$ is the standard form of a separable equation. (Make sure to write the division on LHS as multiplied by the reciprocal .)

- Integrate both sides.

- Finally substitute back $\frac{y}{x}$ in v. Solve for y if possible to get the explicit solution.

Example:

- Solve $(ty + t^2)y' = t^2 + ty + y^2 \quad y(1) = -2$ and $t > 0$.

  Solution:

Recognition: Standard form; $y' = \frac{t^2+ty+y^2}{ty+t^2}$. All terms of numerator and denominator are of total degree 2.

Substitution: Factor t^2 from both numerator and denominator to make the RHS look like a function of $\frac{y}{t}$:

$$y' = \frac{1 + \frac{y}{t} + \frac{y^2}{t^2}}{\frac{y}{t} + 1}$$

$$tv' + v = \frac{1+v+v^2}{v+1}$$

Subtract v: $\quad tv' = \frac{1}{v+1}$

Separable: $(v+1)\,dv = \frac{dt}{t}$

Integrate: $\frac{v^2}{2} + v =$ ~~$\ln|t|$~~ $\ln t$ $+ C$

Implicit Sol.: $v^2 + 2v - 2\ln(t) - 2C = 0$ by substituting back

$$\boxed{\frac{y^2}{t^2} + 2\frac{y}{t} - 2\ln(t) - 2C = 0}$$

Explicit Sol.: $y = t(-1 \pm \sqrt{1 + 2\ln(t) + C}$

IVP: $-2 = -1 - \sqrt{1 + C}$ gives $C = 0$ so

$$\boxed{y = -t - t\sqrt{1 + 2\ln(t)}}$$

- Solve $y' = \frac{t^2+ty+y^2}{t^2}$, $t > 0$.

  Solution:

Recognition: Standard form $y' = \frac{t^2+ty+y^2}{t^2}$. and the total degree of each term is two in the numerator and the denominator.

Substitution: Divide numerator and denominator by t to power the total degree of each term.

$y' = 1 + y/t + (y/t)^2$

Use substitution $v(t) = \frac{y(t)}{t}$. $(y'(t) = tv'(t) + v(t))$.

$v + tv' = 1 + v + v^2$

Subtracting v gives:

$tv' = 1 + v^2$.

This is a separable ode. Applying the method for separable ode we have:

Separable: $(1 + v^2)dv = tdt$

Integrate: $\int \frac{dv}{v^2+1} = \int \frac{dt}{t} + C$

Implicit Sol.: $\arctan(v(t)) = \ln|t| + C$

That is, by substituting back, $\arctan(\frac{y(t)}{t}) = \ln(t) + C$

Explicit Sol.: Taking the tangent of both sides, we have

$\frac{y}{t} = \tan(\ln(t) + C)$

Hence

$$\boxed{y(t) = t\tan(\ln(t) + C)}$$

- $\frac{dy}{dx} = \frac{x+3y}{x-y}$

Solution:

Recognition: $\frac{dy}{dx} = \frac{x+3y}{x-y}$ is homogeneous because the total exponent of each term is one in both numerator and the denominator.

Substitution: Divide both numerator and the denominator by x to power of the total exponent of each term to get the RHS as a function of $\frac{y}{x}$.

$y' = \dfrac{1+3(y/x)}{1-(y/x)}$, then Substitute:

$xv' + v = \frac{1+3v}{1-v} \implies xv' = \dfrac{1+2v+v^2}{1-v}$

Separable: $\frac{1-v}{(1+v)^2}dv = \frac{1}{x}dx$

Integrate: LHS= $\int \left(\dfrac{-1}{1+v} + \dfrac{2}{(1+v)^2}\right)dv$ with a u-sub $u = 1+v$

OR A u-sub $u = 1+v$ and

LHS$= \int \left(\frac{2-u}{u^2}\right) du = \int \left(\frac{2}{u^2} - \frac{1}{u}\right) du$

- $-2(1+v)^{-1} - \ln|v+1| = \ln|x| + C$

Implicit: Substitute back: $\ln(x+y) + \frac{2x}{x+y} + C = 0$

Singular Sol: $y \neq x$

- $\frac{dy}{dx} = -\frac{ax-by}{bx-cy}$ is an homogeneous first order equation. Solving with constant a, b, c is somewhat confusing. I prefer replacing those with numbers.