Methods of solving first order linear odes.

$y' + p(t)y = g(t)$ (Standard form) or

$p_1(t)y' + p_2(t)y = g_1(t)$

Note that you only see one with y and one term with y' on the left and a function of t on the right side of the equation.

- Step 1: Put in the Standard form: $\boxed{y' + p(t)y = g(t)}$
- Step 2: Find the integrating factor: $\boxed{\mu(t) = e^{\int p(t)dt}}$ (Function of t only. Simplify using $e^{a \ln t} = \left(e^{\ln t}\right)^a = t^a$ and $e^{a+b} = e^a.e^b$ if applies.)
- Step 3: Multiply every term of the **Standard form** by $\mu(t)$:
  $\mu(t)y' + \mu(t)p(t)y = \mu(t)g(t)$
- Step 4: Notice that the left hand side is $\mu(t)y' + \mu(t)p(t)y = \left(\mu(t)y\right)'$

  That is :$\left(\mu(t)y\right)' = \mu(t)g(t)$
- Step 5: Integrate : $\mu(t)y = \int \mu(t)g(t)dt + C$ (Note that the integral constant is important here. Also, you may need to simplify$\mu(t)g(t)$ before integrating.)
- Step 6: Find the explicit general solution:

$$y = \frac{1}{\mu(t)}\left(\int \mu(t)g(t)dt\right) + \frac{C}{\mu(t)}$$

(Again note that the integral constant is important since it multiplies reciprocal of a function.

- Step 7: Find the the value of C for the initial value. This step is only needed if there is an initial value.

- Use the following with caution. Best is to use it when integrals don't have a closed form.

$$y = \frac{1}{\mu(t)}\left(\int_{t_0}^{t} \mu(s)g(s)ds\right) + \frac{y_0}{\mu(t)}$$ where $$\mu(t) = e^{\int_{t_0}^{t} p(s)\ ds}$$

A few examples

1. $ty' + 2y = te^{-2t}, \qquad t > 0$

2. $y' + 2y = te^{-t}, \qquad y(1) = 0$

3. $ty' + (t+1)y = t, \qquad y(\ln 2) = 1,\ t > 0$

Solution

Step 1 : $y' + (1 + \frac{1}{t})y = 1$

Step 2 : $\mu(t) = te^{t}$

Step 3 : $te^{t}y' + (t+1)e^{t}y = te^{t}$

Step 4 : $(te^{t}y)' =$ LHS

Step 5 : $te^{t}y = te^{t} - e^{t} + c$

Step 6 : $y = 1 - \dfrac{1}{t} - \dfrac{Ce^{-t}}{t}$

Step 7 : $C = 2$ so $y = 1 - \dfrac{1}{t} - \dfrac{2e^{-t}}{t}$

4. $y' = 2ty + t, \qquad y(0) = 1$

5. $y' = 2ty + 1 \qquad y(2) = 7$

Solution:

Step 1 : $y' - 2ty = 1$

Step 2 : $\mu(t) = e^{-t^2}$

Step 3 : $e^{-t^2}y' - 2te^{-t^2}y = e^{-t^2}$

Step 4 : $(e^{-t^2}y)' =$ LHS ✓

Step 5 : $e^{-t^2}y = \int e^{-t^2}\, dt + C$

Step 6 : $y = e^{-t^2}\left(\int e^{-t^2}\, dt + C\right)$

Step 7 : $y = e^{-t^2}\left(\int_2^t e^{-s^2}\, ds + C\right)$

$$\Longrightarrow 7 = e^{-1^2}\left(\int_2^2 e^{-s^2}\, ds + C\right)$$

$$\Longrightarrow C = 7e$$

$$\Longrightarrow y = e^{-t^2}\left(\int_2^t e^{-s^2}\, ds + 7e\right)$$