Method of solving separable equations

- **Recognition**: If you can not see instantly that an equation is separable, sometimes it helps to solve for $y' = f(t, y)$ and then factor $f(t, y)$ and check if all factors are either functions of t "only" or functions of y "only" (No mixed functions allowed). In this case, $f(t, y) = h(t)g(y)$.

  Generally, multiply both sides of $y' = h(t)g(y)$ by $\dfrac{1}{g(y)}$ to get the standard form.

- Write in the standard form: $\boxed{g(y)dy = f(t)dt}$

- Integrate: $\boxed{\displaystyle\int g(y)dy = \int f(t)dt + C}$

- Use the initial value to solve for C. Make sure to plug the value of C back in the equation.

- The above is an implicit form of the solution.

- You may be able to use algebraic tools to solve the equation **explicitly**. Review your algebraic tool. If you are not able to solve explicitly, keep the implicit form.

  Sometimes to find the explicit solution, you need to solve a **quadratic equation**.

  In case of quadratic equations, you may need to use the **initial values** again to choose between $\pm$ and find the correct curve.

- If the explicit solution was obtained, find the domain of the solution and the find out the curve that goes through the initial value. Also, if a "critical" y value is achieved, exclude it from the domain.

Critical Values:

- Check for **Equilibrium** solutions. That is, where $y' = 0$ so set $g(y) = 0$ and solve. If ode is autonomous, find whether the equilibrium solution is stable or not, using the phase diagram.

- Check for **singular solutions**. The critical values where partial derivative with respect to y doesn't exist or is not continuous.* Specially in this course, where $g(y)$ and hence y' is not defined. †

*Partial derivatives: If you are not and have never been in Calc 3, don't worry, we are going to discuss partial derivatives soon. Follow the instruction that comes next.

†I have not defined the singular solution accurately. The most general definition of a singular solution is a solution that contains points where uniqueness fails. (We will discuss the uniqueness about an initial value soon.) My definitions is adequate for this course. Here we call out all critical solutions that may cause uniqueness to fail.

Examples

- $y' = \dfrac{e^t + t}{y},\ y(1) = -2$

Solution:

Match Notes : $h(t) = e^t + t$ and $g(y) = \frac{1}{y}$

Standard Form : $yy' = e^t + t$

Integrate : $\dfrac{y^2}{2} = e^t + \dfrac{t^2}{2} + c$

Solve Quad. : $y = \pm\sqrt{2e^t + t^2 + c}$

IVP :$c = 3 - 2e$ The curve with the negative.

$y = -\sqrt{2e^t + t^2 + 3 - 2e}$

Domain : Set under the integral equal to zero. Use the calculator to find: $x \simeq .18$ and -1.39.

$t \leq -1.4$ and $t \geq .19$ (Pay attention to rounding so you can stay within the domain.)

Singular sol. : $y \neq 0$

- $y' = \dfrac{e^t + 1}{y}$, $y(1) = 3$

- $y' = \dfrac{3x^2 - 1}{3 + 2y}$ $\quad y(1) = -1$

Solution:

Match notes : $h(x) = 3x^2 - 1$ and $g(y) = 3 + 2y$

Standard Form : $(3 + 2y)y' = (3x^2 - 1)$

Integrate : $3y + y^2 = (x^3 - x) + c$

IVP : $3(-1)^3 + (-1)^2 = (1^3 - 1) + c \implies c = -2$

$y^2 + 3y - (x^3 - x - 2) = 0$ implicit general solution

Explicit Sol. : $y = \frac{-3 \pm \sqrt{9 + 4x^3 - 4x - 8}}{2}$

Right Curve : By the initial solution $+$ is acceptable. (Plug in the initial solution.) $y = \frac{-3 + \sqrt{9 + 4x^3 - 4x - 8}}{2}$

Domain : $4x^3 - 4x + 1 \geq 0$ Setting equal to zero gives: $x \simeq -1.12,\ 0.26,\ 0.84$ (round up, down, up). Use the sign of $4x^3 - 4x + 1 \geq 0$ to find $(-\infty, -1.12] \cup [0.84, \infty)$.

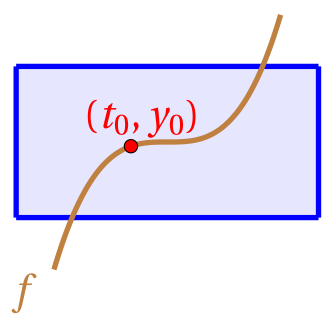
The **domain** is $\boxed{[0.84, \infty)}$ (because of initial value $1 \in [0.84, \infty)$)

Singular Sol. : $y \neq -\frac{3}{2}$

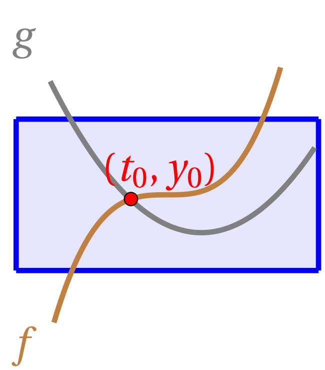
Existence and Uniqueness Theorem. (Picard-Linlelöf)

Consider the ode: $y' = f(t, y)$ and the initial value $y(t_0) = y_0$. Let f be "smooth enough" (continuously differentiable with respect to y.) around (t_0, y_0). Then there is a neighborhood of (t_0, y_0) that contains a curve of solution to the ode that goes through the initial value.

Unique solution through initial value:



Multiple solutions through initial value is a singularity:



Relation to linear first order odes

Notice that this uniqueness conditions when ode is Linear Order Linear Equation $y' + p(t)y = g(t)$ is true if p and g are continuous over a neighborhood of the initial value.

References:

- Boyce, Diprima, and Meade. *Elementary Differential Equations and Boundary Value Problems*