- **Damped forced motion**

  Solve as any nonhomogeneous second order.

Transient solution is the homogeneous part and damping make it vanish over time.

Steady state solution is the particular solution that does not vanish over time.

Resonance: When the forcing function has the same frequency as the natural frequency of the system:

$mu'' + ku = F\cos(\omega_0 t)$ (similarly $mu'' + ku = F\sin(\omega_0 t)$)

Where $\omega_0^2 = \dfrac{k}{m}$

Solution:

$$u_h = C_1\cos(\omega_0 t) + C_2\sin(\omega_0 t)$$

$$u_p = (At)\cos(\omega_0 t) + (Bt)\sin(\omega_0 t)$$

$$u_p' = (A + B\omega_0 t)\cos(\omega_0 t) + (B - A\omega_0 t)\sin(\omega_0 t)$$

$$u_p'' = (-2A\omega_0 - Bt\omega_0^2)\sin(\omega_0 t) + (-At\omega_0^2 + 2B\omega_0)\cos(\omega_0 t)$$

LHS$= (-2mA\omega_0 - Bm\omega_0^2 t + Bkt)\sin(\omega_0 t) + (-At\omega_0^2 m + 2B\omega_0 m + kAt)\cos(\omega_0 t) =$

$-2mA\omega_0 \sin(\omega_0 t) + 2mB\omega_0 \cos(\omega_0 t)$

$B = \frac{F}{2m\omega_0}, \; A = 0$

So $u_p = \frac{Ft}{2m\omega_0}\sin(\omega_0 t)$ The solution is indefinitely increasing.

Or by the method of variation of parameter:

$$u'' + \omega_0^2 u = \frac{F}{m}\cos(\omega_0 t)$$
$$w(\cos(\omega_0 t), \sin(\omega_0 t)) = \omega_0$$

$$\mathbf{u_1} = \int \frac{F}{m\omega_0}\cos(\omega_0 t)\sin(\omega_0 t)\,dt = \int \frac{F\sin(2\omega_0 t)}{2m\omega_0}dt =$$

$$\frac{F}{4m\omega_0^2}\cos(2\omega_0 t) + C_1$$

$$\mathbf{u_2} = \int \frac{F}{2m\omega_0}\cos(\omega_0 t)\cos(\omega_0 t)\,dt =$$

$$\int \left(\frac{F(\cos(2\omega_0 t))}{2m\omega_0} + \frac{F}{2m}\right)dt = \frac{F}{2m\omega_0}\left(\frac{\sin(2\omega_0 t)}{2\omega_0} + t\right) + C_2$$

$$u_p = \frac{F}{4m\omega_0^2}\cos(2\omega_0 t)\cos(\omega_0 t) + \frac{F}{2m\omega_0}\left(\frac{\sin(2\omega_0 t)}{2\omega_0} + \right.$$

$$t\Big)\sin(\omega_0 t) = \frac{F}{4m\omega_0^2}\cos(2\omega_0 t - \omega_0 t) + \frac{F}{2m\omega_0} t\sin(\omega_0 t)$$

$$= \frac{F}{4m\omega_0^2}\cos(\omega_0 t) + \frac{F}{2m\omega_0} t\sin(\omega_0 t)$$

In either situations:

$$u = C_1\cos(\omega_0 t) + C_2\sin(\omega_0 t) + \frac{F}{2m\omega_0} t\sin(\omega_0 t).$$

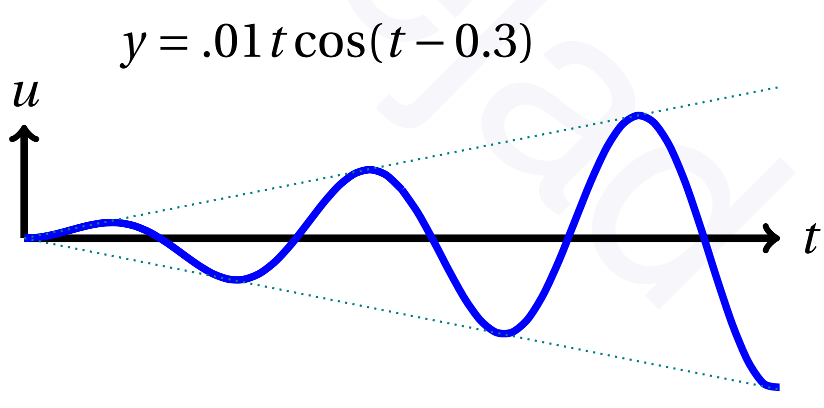
Plug in $u(0) = 0$ and $y'(0) = 0$ to get $u = \frac{F}{2m\omega_0} t\sin(\omega_0 t)$

That is, the solution to system starting at rest looks like:

$u = \frac{Ft}{2m\omega_0}\sin(\omega_0 t)$ Which is a resonance.

Note that the solution is indefinitely increasing.

Example:



Beat $mu'' + ku = F\cos(\omega t)$. Where $\omega \neq \omega_0 = \sqrt{\frac{k}{m}}$ (ω_0 is the natural frequency of the system which is different from the frequency of the force.)

Solution:

$$\omega_0^2 = \frac{k}{m}$$

$$u_h = C_1\cos(\omega_0 t) + C_2\sin(\omega_0 t)$$

$$u_p = A\cos(\omega t) + B\sin(\omega t)$$

$$u_p' = -A\omega\sin(\omega t) + B\omega\cos(\omega t)$$

$$u_p'' = -A\omega^2\cos(\omega t) - B\omega^2\sin(\omega t)$$

LHS= $(-Am\omega^2 + Ak)\cos(\omega t) + (-Bm\omega^2 + Bk)\sin(\omega t)$

$$A = \frac{F}{m(\omega_0^2 - \omega^2)}$$

$$u_p = \frac{F}{m(\omega_0^2 - \omega^2)}\cos(\omega t)$$

If we assume that the initially the mass is at rest (the energy of the system comes entirely from the external force.) $u(0) = 0$ and $u'(0) = 0$

Then $u = \dfrac{F}{m(\omega_0^2 - \omega^2)}\Big(\cos(\omega t) - \cos(\omega_0 t)\Big)$

Use the trig identity:

$$\cos(\omega t) - \cos(\omega_0 t) = 2\sin\frac{(\omega_0 - \omega)t}{2}\sin\frac{(\omega_0 + \omega)t}{2}$$

$$u = \frac{2F}{m(\omega_0^2 - \omega^2)} sin \frac{(\omega_0 - \omega)t}{2}\sin\frac{(\omega_0 + \omega)t}{2}$$

The enveloping function is the sine of differences.

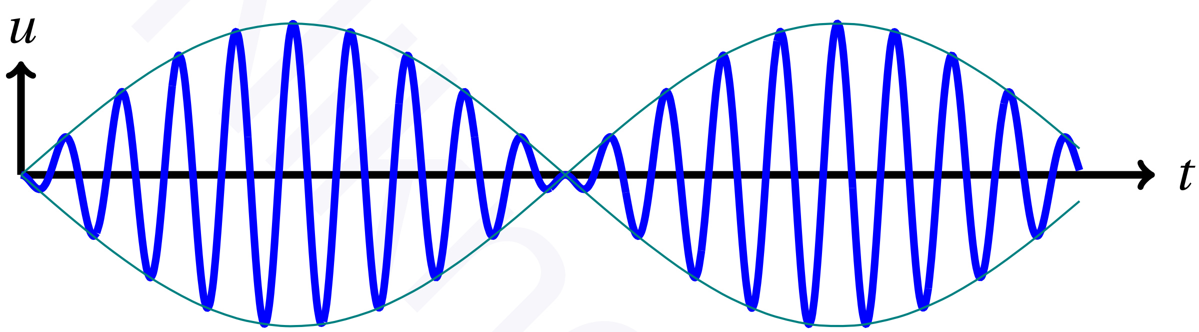
$$\frac{2F}{m(\omega_0^2 - \omega^2)} sin \frac{(\omega_0 - \omega)t}{2}$$

In this case the solution to the system starting at rest looks like:

$$u = \frac{F}{m(\omega_0^2 - \omega^2)}\Big(\cos(\omega t) - \cos(\omega_0 t)\Big) =$$

$$\frac{2F}{m(\omega_0^2 - \omega^2)} sin\Big(\frac{(\omega_0 - \omega)t}{2}\Big) sin\Big(\frac{(\omega_0 + \omega)t}{2}\Big)$$

In this phenomena the amplitude changes periodically. This is used in am radio (**amplitude modulation**).* It is only interesting when ω and ω_0 are “close" in value.



*That is, the signal transfers using amplitude modulation.