Euler-Cauchy ODE

The general form of a homogeneous Euler ode is:

$$y'' + \frac{p}{t}y' + \frac{q}{t^2} = 0$$

where p and q are constants. The coefficients of y' and y are discontinuous at $t = 0$. So we restrict the solution to $t > 0$ or $t < 0$.

The Process:

Note that this is one of those examples which has $y(v(t))$.

Use a new independent variable $v = \ln|t|$

$$\frac{dy}{dt} = \frac{dy}{dv}\frac{dv}{dt} = \frac{1}{t}\frac{dy}{dv}$$

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{1}{t}\frac{dy}{dv}\right) = \left(\frac{1}{t}\frac{d^2y}{dv^2}\frac{dv}{dt} - \frac{1}{t^2}\frac{dy}{dv}\right) = \frac{1}{t^2}\left(\frac{d^2y}{dv^2} - \frac{dy}{dv}\right)$$

That is, $\frac{1}{t^2}\left(\frac{d^2y}{dv^2}+(p-1)\frac{dy}{dv}+qy(v)\right)=0$

$y''+(p-1)y'+q=0$

(Here the independent variable is v.)

Solve this equation and remember that the solution is with respect to v. Replace v by $\ln|t|$.

Example:

Solve $t^2 y'' - 5ty' + 13y = 0$

Solution:

The equation becomes : $\dfrac{d^2y}{dv^2} - 6\dfrac{dy}{dv} + 13y = 0$

The roots are $3 \pm 2i$

so the solution is $y(v) = C_1 e^{3v}\cos(2v) + C_2 e^{3v}\sin(2v)$

$$y(t) = C_1 e^{3\ln|t|}\cos(2\ln|t|) + C_2 e^{3\ln|t|}\sin(2\ln|t|)$$

$$y(t) = C_1|t|^3\cos(2\ln|t|) + C_2|t|^3\sin(2\ln|t|)$$

Example:

$$t^2 y'' - 2ty' + 2y = t^3, \quad t > 0.$$

Solution :

- Change of variable gives $v = \ln|t|$.

- New homogeneous equation is $y'' - 3y' + 2y = 0$ which gives $y_h = C_1 e^{\ln(t)} + C_2 e^{2\ln(t)}$

- Homogeneous Solution: $y_h = C_1 t + C_2 t^2$

- $W(t) = t^2$

- $v_1 = \dfrac{t^2}{2}$ and $v_2 = t$

- Use variation of parameters : $y_p = -\frac{t^3}{2} + t^3 = \frac{t^3}{2}$

- General solution: $y = C_1 t + C_2 t^2 + \frac{t^3}{2}$

Euler-Cauchy ODEs

- The general form of a homogeneous Euler ode is:

  $t^2 y'' + pty' + qy = 0$ or $y'' + \frac{p}{t} y' + \frac{q}{t^2} y = 0$

  where p and q are constants. We often assume $t > 0$ to preserve the continuity of the coefficients.

- Choose $\boxed{v = \ln|t|}$

- Rewrite the equation with v as the independent variable:

  $$\boxed{\frac{d^2 y}{dv^2} + (p-1)\frac{dy}{dv} + qy(v) = 0}$$

- That is solve $y'' + (p-1)y' + qy = 0$ and instead of t in the homogeneous solution, put in v. and then replace v by its value $\ln|t|$ and simplify if possible.

- That is, the characteristic equation is going to be $r^2 + (p-1)r + q = 0$

  There will be 3 cases involving r:

  1. For two distinct roots: r_1 and r_2:

     $y(v) = C_1 e^{r_1 v} + C_2 e^{r_2 v}$

     Replace v by $\ln|t|$ to get: $y(t) = C_1 e^{r_1 \ln|t|} + C_2 e^{r_2 \ln|t|}$

     Simplify to get: $C_1 |t|^{r_1} + C_2 |t|^{r_2}$

  2. For the repeated root r:

     $y(v) = C_1 e^{rv} + C_2 v e^{rv}$

     Replace v by $\ln t$ to get: $y(t) = C_1 e^{r \ln|t|} + C_2 (\ln|t|) e^{r \ln|t|}$

     Simplify to get: $C_1 |t|^r + C_2 |t|^r \ln|t|$

  3. For the complex conjugate roots $r = \lambda \pm \omega i$:

     $y(v) = C_1 e^{\lambda v} \cos(\omega v) + C_2 e^{\lambda v} \sin(\omega v)$

     Replace v by $\ln|t|$ to get: $y(t) = C_1 e^{\lambda \ln|t|} \cos(\omega \ln|t|) + C_2 e^{\lambda \ln|t|} \sin(\omega \ln|t|)$

Simplify to get: $\boxed{C_1|t|^{\lambda}\cos(\omega\ln|t|)+C_2|t|^{\lambda}\sin(\omega\ln|t|)}$