Example:

$t^2y'' - 2y = 3t^2 - 1$ and $t > 0$ and $y_1 = t^2$ and $y_2 = t^{-1}$

Solution

- $y_h = C_1t^2 + C_2t^{-1}$ (We already solved in other notes)
- $y_1 = t^2$ and $y_2 = t^{-1}$
- $$W(t^2, t^{-1}) = \begin{vmatrix} t^2 & t^{-1} \\ 2t & -t^{-2} \end{vmatrix} = -3$$
- Rewrite in the standard form: $y'' + p(t)y' + q(t) = f(t)$ by dividing by the coefficient of y'', that is t^2:

  $y'' - 2t^{-2}y = (3t^2 - 1)(t^{-2})$

  Rewrite the right hand side:

  $\boxed{y'' - 2t^{-2}y = 3 - t^{-2}}$

  Now we can use the variation of parameter formula, knowing that $f(t) = 3 - t^{-2}$:
- $u_1 = -\int \frac{(t^{-1})(3 - t^{-2})}{-3}dt = \int (t^{-1} - \frac{t^{-3}}{3})dt = \ln t + \frac{1}{6}t^{-2} + C$
- $u_2 = \int \frac{(t^2)(3 - t^{-2})}{-3}dt = -\int (t^2 + \frac{1}{3})dt = -\frac{t^3}{3} + \frac{t}{3} + C$
- $u_1y_1 + u_2y_2 = (\ln t + \frac{t^{-2}}{6})t^2 + (-\frac{t^3}{3} + \frac{t}{3})t^{-1} = t^2 \ln t + \frac{1}{6} - \frac{t^2}{3} + \frac{1}{3} = t^2 \ln(t) + \frac{1}{2} - \cancel{\frac{t^2}{3}}$ (*homogeneous*)

  so

  $y_p = t^2 \ln t - \frac{1}{2}$
- $\boxed{y = C_1t^2 + C_2t^{-1} + t^2 \ln t - \frac{1}{2}}$