Procedure for Solving Linear Second-Order ODE

We often consider $y'' + p(t)y' + q(t)y = f(t)$ and we call the function on the right hand side, the **forcing function**.

The procedure for non-homogeneous solving linear second-order ode has two steps:

1. Find the general solution of the homogeneous problem:

   $$y'' + p(t)y' + q(t)y = 0$$

   According to the theory for linear differential equations, the general solution of the homogeneous problem is

   $$y_h(t) = C_1 y_1(t) + C_2 y_2(t)$$

   where C_1 and C_2 are constants and y_1 and y_2 are any two linearly independent solutions to the homogeneous equation.

2. Find a particular solution of the non-homogeneous problem:

   $$y'' + p(t)y' + q(t)y = f(t)$$

   The particular solution is any solution of the non-homogeneous problem and is denoted $y_p(t)$.

The general solution of the non-homogeneous problem is

$$\boxed{y(t) = C_1 y_1(t) + C_2 y_2(t) + y_p(t)}$$

The key point to note is that **all** possible solutions to a linear second-order ode can be obtained from two linearly independent solutions to the homogeneous problem and any particular solution.

Here is an example. Consider the ode

$$y'' + 3y' + 2y = 6e^t$$

The homogeneous equation is

$$y'' + 3y' + 2y = 0$$

Solve for homogeneous solution to get the fundamental set of $y_1 = e^{-t}$ and $y_2 = e^{-2t}$ are solutions to the homogeneous equation.

A particular solution of the non-homogeneous equation is e^t.* Hence, the general solution of the ode is

$$y(t) = C_1 e^{-t} + C_2 e^{-2t} + e^t$$

where C_1 and C_2 are constants.

*We show this later.

Note: The difference between two particular solutions of $y'' + p(t)y' + q(t)y = f(t)$ is a homogeneous solution.

Proof:

Let y_{p1} and y_{p2} be two particular solutions of the ode. Then plug in $(y_{p1} - y_{p2})$ in the LHS:

LHS $= (y_{p1} - y_{p2})'' + p(t)(y_{p1} - y_{p2})' + q(t)(y_{p1} - y_{p2})$

$= (y''_{p1} + p(t)y'_{p1} + q(t)y_{p1}) - (y''_{p2} + p(t)y'_{p2} + q(t)y_{p2})$

$= f(t) - f(t) = 0$ ✓ $\square$

Any solution that satisfies the non-homogeneous solution is called a **particular solution**.*

*This could be any IVP solution to the equation.