Method of undetermined coefficients

To find the solution of $y'' + by' + cy = f(t)$ if $f(t)$ is one of the following:

- $f(t) = e^{at}$
- $f(t) =$ polynomial
- $f(t) = \sin$ or $\cos$
- $f(t) =$ product of two or three of the above terms
- $f(t) =$ sum of various terms that can be obtained from the above.

In this method we make educated guesses for a particular solution $\boxed{y_p}$ with undetermined coefficients and then we find $\boxed{y_p'}$ and $\boxed{y_p''}$. Last, we plug in those values in the LHS and set it equal to RHS.

Then we match the coefficient of the same function on the LHS and RHS

How do we make a guess

- Solve the homogeneous problem and find the homogeneous solution y_{h1} and y_{h2}.

- Make an initial guess: that is
  - For $e^{\text{A Power}}$ Guess $Ae^{\text{That power}}$
  - For $\sin()$ or $\cos()$ Guess $A\cos() + B\sin()$
  - For a polynomial of size n (such as t, t^2, $t^2 + 4$,...), Guess a polynomial of size n, that is, $A_1 x^n + A_2 x^{n-1} + ... + A_n$.
  - For a **sum** of these or **product** of these, Guess **sum** of the guesses or **product** of guesses.

- If any of your guesses is/are the homogeneous solutions y_{h1} or y_{h2}, multiply by t. (Modifies guesses)

- If after multiplying by t still one or more of your guesses is the homogeneous solutions y_{h1} or y_{h2}, multiply by another t (total of t^2). (Modify more)

- **Forcing function is exponential.**

Example:

Solve $y'' + 3y' + 2y = e^{3t}, \quad y(0) = 0, \quad y'(0) = 0.$

Solution:

A homogeneous solution's fundamental set is $\{e^{-t}, e^{-2t}\}$.

We guess that the particular solution has the form $y_p(t) = Ae^{3t}$. Here A is an unknown constant. Compare to the homogeneous solution and notice no modification is needed. In this case we have $y_p' = 3Ae^{3t}$ and $y_p'' = 9Ae^{3t}$. Substituting the expressions for y_p, y_p', and y_p'' into the differential equation, we obtain

$$9Ae^{3t} + 3(Ae^{3t}) + 2Ae^{3t} = e^{3t}$$

The LHS of this last expression is $20Ae^{3t}$. Comparing the two sides of the ode, we find $A = 1/20$.

So the general solution is $y = C_1 e^{-t} + C_2 e^{-2t} + \frac{1}{20}e^{3t}$

plug in the initial value to get $C_1 + C_2 + \frac{1}{20} = 0$.

Take the derivative to get $y' = -C_1 e^{-1} - 2C_2 e^{-2t} + \frac{3}{20} e^{3t}$ and plug in the initial value $y'(0) = 0$ to get:

$-C_1 - 2C_2 + \frac{3}{20} = 0$

Find $C_1 = -.25$ and $C_2 = .2$

$$\boxed{y = -.25e^{-t} + .2e^{-2t} + \frac{1}{20}e^{3t}}$$

- $f(t) =$ **polynomial**

Example:

$$y'' + y' - 2y = t + 1$$

Solution:

A homogeneous solution's fundamental set is: $\{e^t, e^{-2t}\}$.

We guess that the particular solution has the form $y_p(t) = At + B$, where A and B are unknowns. (The guess is not linearly dependent on the homogeneous solutions so no modification is needed.) In this case we have $y_p' = A$ and $y_p'' = 0$. Substituting the expressions for y_p, y_p', and y_p'' into the differential equation, we obtain

$$0 + A - 2(At + B) = t + 1$$

Rearranging the LHS of this equation, we obtain

$$-2At + (A - 2B) = t + 1$$

Comparing the coefficients of t on both sides of the equation, we conclude that $-2A = 1$. Comparing the constant terms, we conclude that $A - 2B = 1$. The first equation implies $A = -1/2$. This fact and the second equation imply that $B = -3/4$

$$y_p = -0.5t - 0.75$$

and

General solution is $y = C_1 e^t + C_2 e^{-2t} - 0.5t - 0.75$

Note: If the non-homogeneous term is a polynomial of degree n, then an initial guess for the particular solution should be a polynomial of degree n:

$$y_p = A_n t^n + A_{n-1} t^{n-1} + A_{n-2} t^{n-2} + ... A_1 t + A_0$$

This guess may need to be modified.

- **Forcing term** $f(t) = \sin$ **or** $\cos$:

Example:

$$y'' + y' - 2y = \sin(3t)$$

Solution:

We guess that the particular solution is

$y_p = A\cos(3t) + B\sin(3t)$. Note that we have both sine and cosine terms and they are linearly independent of the homogeneous solutions. In this case we have

$$y_p' = -3A\sin(3t) + 3B\cos(3t) \qquad y_p'' = -9A\cos(3t) - 9B\sin(3t)$$

Substituting y_p, y_p', and y_p'' into the differential equation we obtain:

$$(-9A + 3B - 2A)\cos(3t) + (-9B - 3A - 2B)\sin(3t) = \sin(3t)$$

Rearranging the LHS of this equation, we obtain:

$(-11A+3B)\cos(3t)+(-11B-3A)\sin(3t)=\sin(3t)$

Comparing the coefficients of $\sin(3t)$ on both sides of the equation, we conclude $-11B-3A=1$. The RHS has no cosine terms. This means that the coefficient of cosine on the LHS must be 0 - That is, $3B-11A=0$. Solving these equations we conclude $A=-\frac{3}{130}$ and $B=-\frac{11}{130}$. So $y_p=-\dfrac{3}{130}\cos(3t)-\dfrac{11}{130}\sin(3t)$

and general solution:

$$\boxed{y=C_1e^t+C_2te^{-2t}-\frac{3}{130}\cos(3t)-\frac{11}{130}\sin(3t).}$$

Note: In general if the non-homogeneous term is of the form $A\cos(ct)+B\sin(ct)$. Then an initial guess for the particular solution is $y_p=A\cos(ct)+B\sin(ct)$.

This guess may need to be modified.

Regarding the example, you may want to try to guess $y_p=A\sin(3t)$, then $y_p'=3A\cos(3t)$ and $y_p''=-9A\sin(3t)$. After plugging in the solutions, we get $(-11A)\sin(3t)+3A\cos(3t)=\sin(3t)$. That is, $A=-1/11$ and $B=0$, an inconsistent solution. That is when we know that an incorrect solution was chosen.

- $f(t) =$ **Sum of various terms**

Example: Consider the differential equation

$$y'' + y' - 2y = t + 1 + \sin(3t)$$

Solution:

If the non-homogeneous term is a sum of two terms, then the particular solution is $y_p = y_{p_1} + y_{p_2}$, where y_{p_1} is a particular solution of $y'' + y' - 2y = t + 1$ and y_{p_2} is a particular solution $y'' + y' - 2y = \sin(3t)$. From our discussion above, we know $y_{p_1} = -0.5t - 0.75$ and that $y_{p_2} = -\frac{3}{130}\cos(3t) - \frac{11}{130}\sin(3t)$. Hence the complete particular solution is $y_p = -0.5t - 0.75 - \frac{3}{130}\cos(3t) - \frac{11}{130}\sin(3t)$.

General solution is:

$$\boxed{y = C_1 e^{t} + C_2 e^{-2t} - 0.5t - 0.75 - \frac{3}{130}\cos(3t) - \frac{11}{130}\sin(3t)}$$

Remember the initial guesses above may need to be modified.

- $f(t) =$ **Product of various factors**

Example:

$y'' + y' - 2y = 130t\sin(3t)$

Solution:

Note: The guess for each factor multiply.

The guess for this one is $y_p = (At+B)(C\cos(3t)+D\sin(3t))$. We do not need complicated coefficients.* So guess $y_p = (At+B)\cos(3t)+(Ct+D)\sin(3t)$. None of the terms will be linearly dependent of homogeneous solution. So no modification is needed.

$y_p' = A\cos(3t) - 3(At+B)\sin(3t) + C\sin(3) + 3(Ct+D)\cos(3t)$
and

$y_p'' = -3A\sin(3t) - 3A\sin(3t) - 9(At+B)\cos(3t) + 3C\cos(3t) + 3C\cos(3t) - 9(Ct+D)\sin(3t) = (6C-9B)\cos(3t) - 9At\cos(3t) + (-6A-9D)\sin(3t) - 9Ct\sin(3t)$

*I explain this in class.

Plugging in the left hand side of the equation and re-grouping according to the function type will result in:

$$(6C-9B+A+3D-2B)\cos(3t)+(-6A-9D+C-3B-2D)\sin(3t)+(-9A+3C-2A)\,t\cos(3t)+(-9C-3A-2C)\,t\sin(3t)=130\,t\sin(3t).$$

Therefore,

$$\begin{cases} A-11B+6C+3D=0 \\ -6A-3B+C-11D=0 \\ -11A+3C=0 \\ -3A-11C=130 \end{cases}$$

Use your calculator to solve:

$$A=-3,\ B=-\frac{369}{65},\ C=-11,\ D=\frac{142}{65}.$$

The particular solution is

$$y_p=-3t\cos(3t)-\frac{369}{65}\cos(3t)-11t\sin(3t)+\frac{142}{65}\sin(3t).$$

The general solution is

$$y=C_1e^t+C_2e^{-2t}-3t\cos(3t)-\frac{369}{65}\cos(3t)-11t\sin(3t)+\frac{142}{65}\sin(3t)$$

- **Modified Guesses**

Consider the example:

$$y'' + y' - 2y = e^t$$

Solution:

Wrong Guess: According to the previous arguments, the guess for the particular solution is $y_p = Ae^t$. Then $y_p' = Ae^t$ and $y_p'' = Ae^t$. Substituting these expressions into the ode, we have

$$Ae^t + Ae^t - 2Ae^t = e^t$$

The LHS of this expression equals 0, An inconsistent equation!

This means that the initial guess $y_p = Ae^t$ is not correct. The reason is that e^t is a solution to the homogeneous equation $y'' + y' - 2y = 0$.

The appropriate guess in this case is $y_p = Ate^t$. We have $y_p = Ae^t - Ate^t$ and $y_p'' = 2Ae^t + Ate^t$. Substituting into the ode we have,

$$Ate^t + 2Ae^t + Ae^t + Ate^t - 2Ate^t = e^t$$

Cleaning up the LHS, we have $3Ae^t = e^t$. Notice that the function te^t simplified. Hence, $A = 1/2$.

Replace $\frac{1}{3}$ for A in the guess: $y_p = \frac{1}{3}te^t$ and the general solution is $y = C_1 e^t + C_2 e^{-2t} + \frac{1}{3}te^t$

We conclude that we must multiply an initial guess for the particular solution by t^n, where n is the smallest integer so that no **term** in the modified particular solution is a solution to the homogeneous ode. The next example explains when $n > 1$.

Example:

$y'' + 2y' + y = e^{-t}$.

Solution:

The characteristic equation has a repeated root. The fundamental set of homogeneous solutions are $\{e^{-t}, te^{-t}\}$

As an initial guess, $y_p = Ae^{-t}$. The modified guess is $y_p = At^2e^{-t}$. We need to multiply by t^2 since te^{-t} is also a solution to the homogeneous ode.

Now $y_p' = -At^2e^{-t} + 2Ate^{-t}$ and

$y_p'' = At^2e^{-t} - 4Ate^{-t} + 2Ae^{-t}$.

Plug in the left hand side:

$At^2e^{-t} - 4Ate^{-t} + 2Ae^{-t} - 2At^2e^{-t} + 4Ate^{-t} + At^2e^{-t} = e^{-t}$

Notice that terms with te^{-t} and t^2e^{-t} simplify and what is remaining is:

$2Ae^{-t} = e^{-t}$ and $A = 0.5$

$y_p = 0.5t^2e^{-t}$ and the general solution is

$y = C_1e^{-t} + C_2te^{-t} + 0.5t^2e^{-t}$

Example:

$y'' - 3y' = t,$

Solution:

The fundamental set of homogeneous solution is $\{e^{3t}, 1\}$

The initial guess would be $y_p = At + B$. The term B, a constant is a solution to the homogeneous part. Hence, the modified guess is $y_p = At^2 + Bt$. Find the derivatives: $y_p' = 2At + B$ and $y_p'' = 2A$.

LHS= $A - 2At - B = t$ gives $A - B = 0$ and $-2A = 1$ so $A = -0.5$ and $B = 0.5$.

$y_p = -0.5t^2 + 0.5t$ and general solution:

$y = C_1 + C_2 e^{3t} - 0.5t^2 + 0.5t$

Process:

- Find the homogeneous fundamental set.
- Make an initial guess for all functions in each term and factors.
- Compare to fundamental set and multiply with t, until each term is linearly independent of the fundamental set.
- Take derivatives and plug in.
- Regroup all terms with the same functions.
- Equate the coefficients of the same function in the left and right and side of the equation. Then solve for the coefficients.
- Plug the coefficients back in the modified guess. Add the homogeneous solution to get the general solution.