Solving Homogeneous 2nd order linear odes with constant coefficients.

Summary of Theorems:

For enough continuity within the coefficients, every homogeneous 2nd order linear ode, if we find two linearly independent solutions, y_1 and y_2, then we can find the unique solution for every ivp by finding the coefficients of $C_1 y_1 + C_2 y_2$.

We call $\{y_1, y_2\}$ is called the fundamental set of solutions for the ode.

Finding the two linearly independent solutions is possible using different methods:

- Reduction of order.

- Series.

One or both will be assigned as homework problems.

Constant-Coefficient Homogeneous ODE

A constant-coefficient homogeneous second-order ode is of the form

$$ay''(t) + by'(t) + cy(t) = 0$$

where a, b and c are constants. Recall that the **general solution** is

$$y(t) = C_1 y_1(t) + C_2 y_2(t)$$

where C_1 and C_2 are constants and $y_1(t)$ and $y_2(t)$ are any two linearly independent solutions of the ode. Through methods that we have mentioned, we get the following results for the solutions.

Definition:

Consider the equation $ay'' + by' + cy = 0$.

The **characteristic** equation of the equation is the equation $\boxed{ar^2 + br + c = 0}$.

The solution to ode depends on the roots of the characteristic equation.

Distinct Roots:

From the methods mentioned before, we know that the solution looks like: $y = e^{rt}$. Where r is a constant. To find r, plug in e^{rt} in the ode. If $y = e^{rt}$, then $y' = r(e^{rt})$ and $y'' = r^2(e^{rt})$. Substituting these into the ode, we have

$$ar^2e^{rt} + bre^{rt} + ce^{rt} = e^{rt}(ar^2 + br + c) = 0$$

This equation is satisfied if $e^{rt} = 0$ or $r^2 + pr + q = 0$. But $e^{rt} \neq 0$.

Hence, r must satisfy the equation

$ar^2 + br + c = 0$ which is the characteristic equation of the ode. That is, r is a root for characteristic equation.

Repeated Roots

Now if we repeated roots for r, then $y = te^{rt}$ is additionally a solution of the equation:

$y' = e^{rt} + rte^{rt}$

$y'' = 2re^{rt} + r^2te^{rt}$

So $a(2re^{rt} + r^2te^{rt}) + b(e^{rt} + rte^{rt}) + cte^{rt} = 0$

Regroup the terms: $te^{rt}(ar^2 + br + c) + (2are^{rt} + be^{rt}) = 0 + 2ae^{rt}(r + \frac{b}{2a}) = 0$

For repeated roots: $r = -\frac{b}{2a}$.

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*The second solution can be found using a method called variation of parameter. Since e^{rt} is a solution, we may find another solution in the form ve^{rt}. Plug in the ode to get: $av'' + (b + 2ar)v' = 0$' That is $av'' = 0$ That is, $v = t$ which explains the solution.

Complex Conjugate Roots

Suppose that the characteristic polynomial has complex roots $a+ib$ and $a-ib$, where a and b are real. These are distinct roots, so the the general solution can be written:

$$y(t) = C_1 e^{(a+ib)t} + C_2 e^{(a-ib)t}$$

The problem with writing the solution in this form is that it involves complex-valued functions. It is possible to re-express the general solution in terms of two linearly independent real-valued functions.

To be able to do that, we use Euler's Identity.

Euler's identity:

$$\boxed{e^{i\omega} = \cos(\omega) + i\sin(\omega)}$$

Exponential functions were originally defined for real numbers. To be able to define them for complex numbers, the new function was build so the many properties of exponential functions over real domain is preserved.

So the definition was made possible by Taylor series.

The Taylor series for exponential about $x = 0$ is $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. So we define the complex version of exponential function using Taylor's Series.

How to derive Euler identity?

Plug in $x = it$ to get: $e^{it} = \sum_{n=0}^{\infty} \frac{(it)^n}{n!}$

Now remember $i^2 = -1,\ i^3 = -i,\ i^4 = 1, \ldots$

So $e^{it} = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!}$

First one is the Taylor series for $\cos(t)$ and the second one is the Taylor series for $\sin t$.

So $e^{it} = \cos(t) + i\sin(t)$.

Now What is $e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$.

That is,

$$e^{(\lambda + i\omega)t} = e^{\lambda t} e^{i\omega t} = e^{\lambda t}\Big(\cos(\omega t) + i\sin(\omega t)\Big)$$

$$= e^{\lambda t}\cos(\omega t) + i e^{\lambda t}\sin(\omega t)$$

Similarly, we have

$$e^{(\lambda - i\omega)t} = e^{\lambda t} e^{-i\omega t} = e^{\lambda t}\Big(\cos(\omega t) - i\sin(\omega t)\Big)$$

Substituting, these two expressions into the general solution to get:

$$y(t) = C_1 e^{\lambda t}\Big(\cos(\omega t) + i\sin(\omega t)\Big) + C_2 e^{\lambda t}\Big(\cos(\omega t) - i\sin(\omega t)\Big)$$

Choose $C_1 = 0.5$ and $C_2 = 0.5$. This yields the solution:

$y_1(t) = e^{\lambda t} \cos(\omega t)$

Now choose $C_1 = i0.5$ and $C_2 = -i0.5$. This yields the solution:

$y_2(t) = e^{\lambda t} \sin(\omega t)$

Both of these functions are solutions to the original ode. In addition, they are linearly independent, since they are not multiples of each other. So they form a fundamental set of solution for the ode. So we can write:

$y(t) = D_1 e^{\lambda t} \cos(\omega t) + D_2 e^{\lambda t} \sin(\omega t)$ is a general solution to the ode,

where D_1 and D_2 are constant real numbers.

1. **Characteristic Polynomial has Distinct Roots**

Example: Solve

$y'' - 3y' - 18y = 0, \qquad y(0) = 1 \ , y'(0) = 2.$

Solution:

The characteristic polynomial is

$r^2 - 3r - 18 = (r-6)(r+3)$, which has roots $r = 6$ and $r = -3$.

The **general solution** is : $y = C_1 e^{6t} + C_2 e^{-3t}$ and plug in $y(0) = 1$ to get $C_1 + C_2 = 1$.

Take the derivative of the solution: $y' = 6C_1 e^{6t} - 3C_2 e^{-3t}$ and $y'(0) = 2$ to get $6C_1 - 3C_2 = 2$.

Solve $\begin{cases} C_1 + C_2 = 1 \\ 6C_1 - 3C_2 = 2 \end{cases}$

That is $C_1 = \frac{5}{9}$ and $C_2 = \frac{4}{9}$.

IVP: $\boxed{y = \frac{5}{9} e^{6t} + \frac{4}{9} e^{-3t}}$

2. **Characteristic Polynomial has a Double Root**

Example:

Solve $y'' + 6y' + 9y = 0, \qquad y(0) = 1$ and $y'(0) = 2$

Solution:

The characteristic polynomial is $r^2 + 6r + 9 = (r + 3)^2$, which has a double root -3. The general solution is

$y(t) = C_1 e^{-3t} + C_2 t e^{-3t}$ and plug in the initial value $y(0) = 1$ to get $C_1 = 1$.

Take the derivative of the solution: $y'(t) = -3C_1 e^{-3t} - 3C_2 t e^{-3t} + C_2 e^{-3t}$ and plug in the $y'(0) = 2$ to get

$-3C_1 + C_2 = 2$.

$$\begin{cases} C_1 = 1 \\ -3C_1 + C_2 = 2 \end{cases}$$

$C_1 = 1$ and 5.

So the solution is **IVP:** $\boxed{y = e^{-3t} + 5te^{-3t}}$

3. **Complex-Conjugate Roots**

Example:

Solve $y'' + 4y = 0$, $y(0) = 2, y'(0) = 6$.

Solution:

The characteristic equation is: $r^2 + 4 = 0$ which gives solutions $r = \pm 2i$.

The general solution is $y(t) = C_1 \cos(2t) + C_2 \sin(2t)$.

Plug in $y(0) = 2$ to get $C_1 = 2$.

Find the derivative of the solution: $y'(t) = -2C_1 \sin(2t) + 2C_2 \cos(2t)$. Plug in $y'(0) = 6$. Gives $C_2 = 3$.

IVP is: $\boxed{y = 2\cos(2t) + 3\sin(2t)}$

Example:

$y'' - 6y' + 13y = 0, \qquad y(\pi/4) = 1$ and $y'(\pi/4) = 3$

Solution: The characteristic polynomial is

$r^2 - 6r + 13$. Using the quadratic formula, we find that the roots are $3+2i$ and $3-2i$. This gives $y_1 = e^{3t}\cos(2t)$ and $y_2 = e^{3t}\sin(2t)$.

The general solution is:

$y(t) = C_1 e^{3t}\cos(2t) + C_2 e^{3t}\sin(2t)$

Plug in $y(\pi/4) = 1$ to get $C_1 e^{3\pi/4} = 1$ which gives $C_2 = e^{-3\pi/4}$.

The derivative of the solution is:

$$y = 3C_1e^{3t}\cos(2t) - 2C_1e^{3t}\sin(2t) + 3C_2e^{3t}\sin(2x) + 2C_2e^{3t}\cos(2t)$$

plug in $y'(\pi/4) =$: $3C_2e^{3\pi/4} - 2C_1e^{3\pi/4} = 3$

Solve using the value for C_1:

$C_2 = e^{-3\pi/4}$ and $C_1 = 0$.

So the solution is $\boxed{y(t) = e^{-3\pi/4}e^{3t}\sin(2t)}$

Definition:

Consider the equation $ay'' + by' + cy = 0$.

The **characteristic** equation is $\boxed{ar^2 + br + c = 0}$.

Solve the quadratic equation: $r = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Then depending on the type of the roots, one of the following options gives the solution:

- Characteristic polynomial has **distinct roots** r_1 and r_2, then the general solution is $\boxed{y_h = C_1 e^{r_1 t} + C_2 e^{r_2 t}}$

- Characteristic polynomial has a **repeated root** r, then the general solution is $\boxed{y_h = C_1 e^{rt} + C_2 t e^{rt}}$

- Characteristic polynomial has **complex conjugate roots**: $\lambda \pm i\omega$, then the general solution is:

  $$\boxed{y_h = C_1 e^{\lambda t}\cos(\omega t) + C_2 e^{\lambda t}\sin(\omega t)}$$

If the equation is an initial value problem:

$$\boxed{ay'' + by' + cy = 0, \quad y(t_0) = a, \quad y'(t_0) = b}$$

- Plug in the initial value in the general solution. You will end up with an equation of C_1 and C_2.

- Take the first derivative of the general equation and plug in the second initial value.

- You will find a second equation of C_1 and C_2.

- Solve the system for C_1 and C_2. Plug back into the general solution.

Certain type of homogeneous linear second-order ode have closed form solutions. We will consider two classes:

1. Constant-coefficient ode:

   $$y'' + py' + qy = 0$$

   $p(t)$ and $q(t)$ are constants.

2. Euler-Cauchy ode:

   $$y'' + \frac{p}{t}y' + \frac{q}{t^2}y = 0$$

   p and q are constants.

This set of notes explained the first method.