Introduction to Second Order Linear ODEs

Definitions: A linear second-order ODE has the form:

$$A(t)\,y''(t) + P(t)\,y'(t) + Q(t)\,y(t) = F(t)$$

On any interval where $A(t)$ is not equal to 0, the above equation can be divided the entire equation by $A(t)$ to get the **standard form**:*

$$y''(t) + p(t)\,y'(t) + q(t)\,y(t) = f(t)$$

The equation is called **homogeneous**† if $f(t) = 0$. Otherwise, it is called **non-homogeneous** and $f(t)$ is called the forcing function.

*That is, $p(t) = \dfrac{P(t)}{A(t)}$, $q(t) = \dfrac{Q(t)}{A(t)}$ and $f(t) = \dfrac{F(t)}{A(t)}$.

†The homogeneous equation looks like

$$y''(t) + p(t)\,y'(t) + q(t)\,y(t) = 0$$

A second-order differential equation may have **initial conditions** or **boundary conditions**. Initial conditions are in the form: $\begin{cases} \mathbf{y}(t_0) = y_0 \\ \mathbf{y}'(t_0) = y_0' \end{cases}$. Boundary conditions might be of the form: $\begin{cases} y(t_0) = a \\ y(t_1) = b \end{cases}$. In this course, we discuss the initial conditions.

To be able to explain all solutions to a second order ode, we need to discuss a few theorems.

Theorem: (Superposition)

Let y_1 and y_2 be two solutions of $y''+p(t)y'+q(t)y=0$, then $C_1y_1+C_2y_2$ is also a solution to the equation.

y_1 and y_2 are solutions so $\begin{cases} y_1''+p(t)y_1'+q(t)y_1=0 \\ y_2''+p(t)y_2'+q(t)y_2=0 \end{cases}$

Multiply the first row by C_1 and the second row by C_2:

$$\begin{cases} C_1y_1''+p(t)C_1y_1'+q(t)C_1y_1=0 \\ C_2y_2''+p(t)C_2y_2'+q(t)C_2y_2=0 \end{cases}$$

Add both sides and regroup based on the derivative order.

$$(C_1y_1''+C_2y_2'')+p(t)(C_1y_1'+C_2y_2')+q(t)(C_1y_1+C_2y_2)=0.$$

That is, $(C_1y_1+C_2y_2)''+p(t)(C_1y_1+C_2y_2)'+q(t)(C_1y_1+C_2y_2)=0$ The later proves the theorem.

Theorem: (Existence and Uniqueness)

For the initial value problem, $y''+p(t)y'+q(t)y=f(t)$ and $\begin{cases} \mathbf{y}(t_0)=y_0 \\ \mathbf{y}'(t_0)=y_0' \end{cases}$, if $p(t)$, $q(t)$ and $f(t)$ are continuous on some interval (a,b) containing t_0, then there exists a **unique** solution $y(t)$ to the ode on the interval (a,b).

Definition: y_1, y_2, ... y_n are linearly independent if and only if "at least one of $C_1, C_2, ..., C_n$ is none-zero implies $C_1 y_1 + C_2 y_2 + ... + C_n y_n \neq 0$".

We can infer that y_1, y_2 are linearly independent if and only if they are not multiple of each other.

Theorem: (Space of Solutions)

Let y_1 and y_2 be two linearly independent solutions to $y'' + p(t)y' + q(t)y = 0$, on an interval where p and q are continuous, then every solution to the equation is of the form $y = C_1 y_1 + C_2 y_2$.

The set $\{y_1, y_2\}$ is called a **fundamental set** of solutions for the ode.

Summary:

For enough continuity within the coefficients, every homogeneous 2nd order linear ode, if we find two linearly independent solutions, y_1 and y_2, then we can find the unique solution for every ivp by finding the coefficients of $C_1 y_1 + C_2 y_2$.