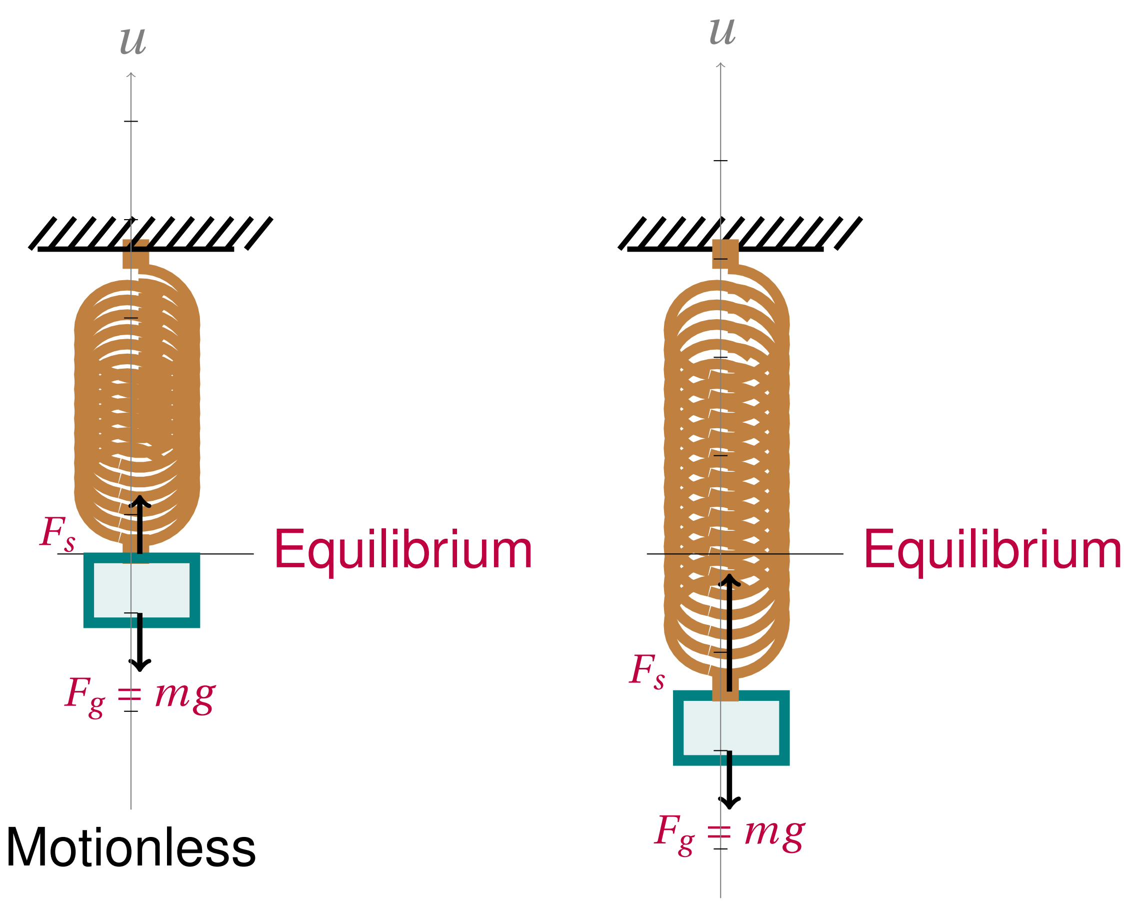
Spring-Mass Systems



Hooks Law The force from the spring is proportional to elongation/contraction of the spring.

Equilibrium is where $kL = mg$. That is where the Hook's force and the gravitational force are neutralized with each other.

Where k is the constant of the hooks Law and L is the **elongation** of the spring when in Equilibrium position.

u is the disposition from the equilibrium position and is a function of time. *

Forces we assume that upward direction is positive. The gravitational force is in the negative direction.

F_s is the spring force. When $L-u>0$, $F_s = k(L-u) > 0$ and when $L-u<0$, the direction of $F_s = k(L-u) < 0$. Note that for $u>0$, the elongation is smaller than L and when $u<0$, the elongation is more than L.

Damping in smaller velocities is proportional to the velocity. $F_d = -\gamma u' > 0$ when $u' < 0$ and F_d and $F_d = -\gamma u' < 0$ when $u' > 0$.

*Most books denote it by x but I saw a book using u and I noticed the last term of the equation has a good ring to it.

Newton law $ma = -mg + k(L-u) - \gamma u' + F_{\text{external}}$

But $kL - mg = 0$ so

$mu'' + \gamma u' + ku = F_{\text{external}}$

Example:

A mass weighing 3 lbf stretches a spring 3 inches. If the mass is pushed downward, stretching the spring by 1 inch and set in motion by a upward velocity of 2 ft/sec and if there is no damping, find the position[†] $u(t)$ at any time t. Determine the frequency, the period and the amplitude of the motion. [‡]

Solution:

First to find the spring constant of proportionality, k, use the formula $k = \frac{mg}{L} = \frac{3}{.25} = 12$.

[†]Position compared to the equilibrium position.

[‡]Note that there is no external force.

No damping means $\gamma = 0$ and $m = \dfrac{w}{g} = \dfrac{3}{32}$

The initial displacement is $u(0) = -1\,in = -\dfrac{1}{12} ft$

and the initial velocity is $u'(0) = 2$.

$$\begin{cases} \dfrac{3}{32}u'' + 12u = 0 \\ u(0) = -\dfrac{1}{12} \\ u'(0) = 2 \end{cases}$$

$u(t) = A\cos(8\sqrt{2}t) + B\sin(8\sqrt{2}t)$

Using initial values: $u(t) = -\dfrac{1}{12}\cos(8\sqrt{2}t) + \dfrac{1}{4\sqrt{2}}\sin(8\sqrt{2}t)$

Oscillation is present to learn more about the oscillation, we find: The angular frequency is $\omega = 8\sqrt{2}$

Amplitude of vibration is $R = \sqrt{\frac{1}{12^2} + \frac{1}{32}} = \sqrt{\frac{11}{288}} \simeq .195\text{ft}$

Period: $T = \frac{2\pi}{\omega} = \frac{2\pi}{8\sqrt{2}}$ Phase: $\delta = \arctan\left(-\frac{\frac{1}{4\sqrt{2}}}{\sqrt{\frac{11}{288}}}\right) + \pi \simeq$
$-1.3 + 3.14 = 2.01$ rad. So $u = \sqrt{\frac{11}{288}}\cos(8\sqrt{2}t - 2.01)$

The free mechanical vibrations:[*]

Let $\begin{cases} mu'' + \gamma u' + ku = 0 \\ u(0) = u_0 \\ u'(0) = u'_0 \end{cases}$ be the initial value problem for a spring.

Where m is the mass, γ the damping constant and k is the spring constant.

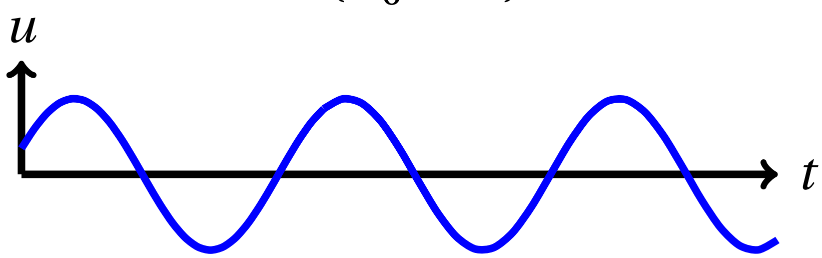
Also $k = \dfrac{mg}{L}$ where L is the elongation of the spring in equilibrium position.

There are two cases involving γ:

1. $\gamma = 0$ The system is **undamped**.

[*]When the forcing function is zero and the vibration solely depends on the initial values of the system. If damping exists, then the solution asymptotically goes to zero as $t \to \infty$.

$u = A\cos(\omega_0 t - \delta)$

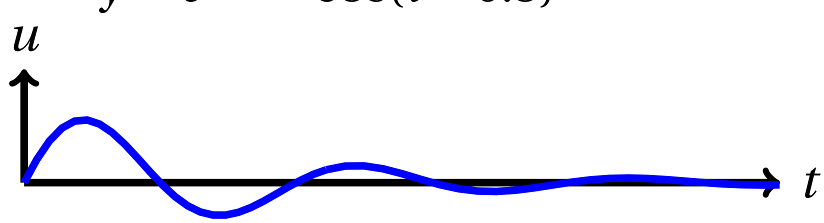


2. $\gamma \neq 0$ then the system is **damped**.

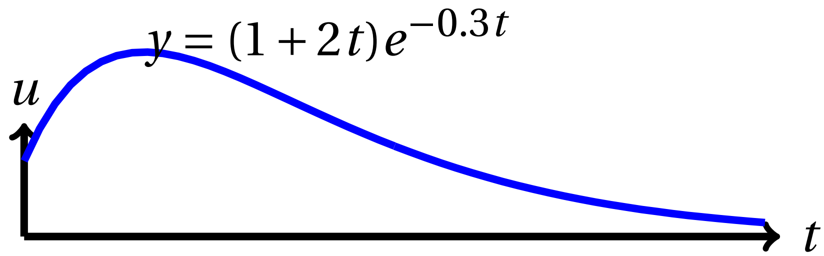
   Solve the initial value problem and notice that regarding the damping there are three cases:

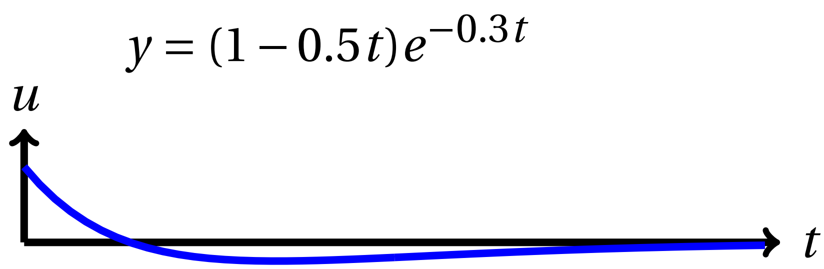
   (a) If $mr^2 + \gamma r + k = 0$ has complex conjugate roots then system is **under-damped**.

   $y = e^{-0.3t}\cos(t - 0.3)$

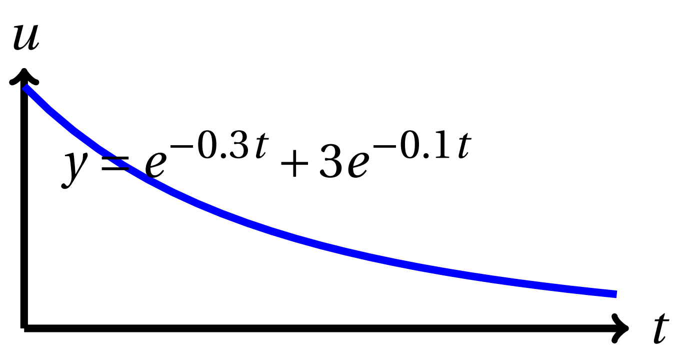
   

   (b) If $mr^2 + \gamma r + k = 0$ has repeated roots solution then system is **critically damped**.





(c) If $mr^2 + \gamma r + k = 0$ has two distinct roots then system is **over-damped**.



Now lets describe the movement of the spring in both damped and undamped cases:

- **Undamped Free Vibrations**†

  Free means there is no external force and we assume the spring to be an ideal undamped spring, then the differential equation becomes

  $mu'' + ku = 0$

  Since m and k are both positive, we get the complex roots

  $mr^2 + k = 0$

  Which has complex roots with $\lambda = 0$ and $\mu^2 = k/m$

  $\omega_0^2 = k/m$. We call $\omega_0 = \sqrt{k/m}$ the **natural frequency** of the system.

  In particular, the solution for the undamped free spring is

  $u = A\cos(\omega_0 t) + B\sin(\omega_0 t)$

  The **natural frequency** of the system is $\boxed{\omega_0 = \sqrt{\frac{k}{m}}}$

†Undamped is a limit of small damping as well.

Note that the frequency here is the angular frequency and is different from the frequency of vibration in 1 unit of time.

The period of the motion:

This is periodic motion with period

$$\boxed{T = \frac{2\pi}{\omega_0}}$$

The motion can be written as $\boxed{u = R\cos(\omega_0 t - \delta)}$.

The maximum displacement R of the mass from equilibrium is the **amplitude** of the motion:

$$\boxed{R = \sqrt{A^2 + B^2}}$$

The parameter δ is called the **Phase** or phase angle and measures the displacement of the wave from $\cos(\omega_0 t)$.

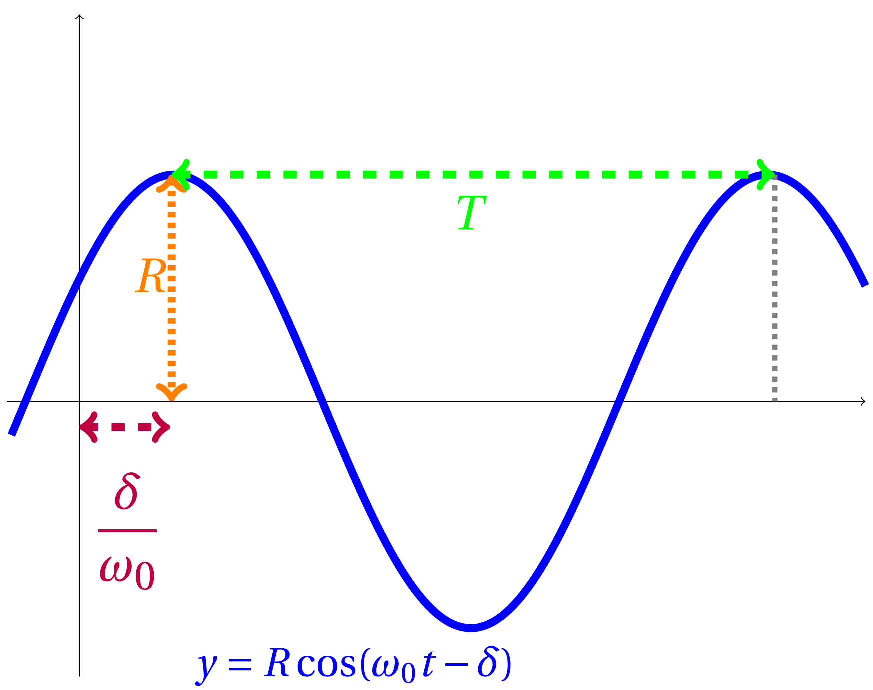
$$\boxed{\tan\delta = \frac{B}{A}}$$

Since the equation of the motion is $\boxed{u = R\cos(\omega_0 t - \delta)}$, *the spring is vibrating with the same amplitude forever.*

Caution: Choose the correct quadrant: sign of sine should match sign of B. Sign of cosine should match sign of A.

That is,

If $A > 0$, then $\boxed{\delta = \arctan \frac{B}{A}}$. If $A < 0$, then $\boxed{\delta = \arctan \frac{B}{A} \pm \pi}$.



- **Damped Free Motion**

  If we do not ignore friction, then we get damped free motion. The equation becomes

  $mu'' + \gamma u' + ku = 0$

  The characteristic equation is

  $m^2 + \gamma r + k = 0$

  The solution depends on the sign of $\gamma^2 - 4km$. We have the following three cases

  1. $\gamma^2 - 4km < 0$: The solution is

     $u = e^{-\gamma t/2m}(A\cos(\mu t) + B\sin(\mu t))$

     Where $\boxed{\mu = \sqrt{\dfrac{4km - \gamma^2}{4m^2}}}$ is the **quasi frequency**.

     In this case the spring vibrates forever but the magnitude of vibration gets closer and closer to 0 in time (Goes to zero as $t \to \infty$)

  2. $\gamma^2 - 4km = 0$: The solution is

$u = (A + Bt)e^{-\gamma t/2m}$

In this case the spring reacts in the beginning if the initial values are nonzero and then it gets closer and closer to equilibrium " asymptotically" as time passes (Goes to zero as $t \to \infty$)

3. $\gamma^2 - 4km > 0$: The solution is

   $u = Ae^{r_2 t} + Be^{r_1 t}$

   Where r_1 and r_2 are the real roots of the characteristic equation. Notice that these roots are both negative since

   $\gamma^2 - 4km < \gamma^2$

   In this case the spring does not even react in the beginning. If the initial values are nonzero, *it gets closer and closer to equilibrium "asymptotically" as time passes. (Goes to zero as* $t \to \infty$)

Briefly, motion of **a damped system** can fall into one of the following cases.

Let $\begin{cases} mu'' + \gamma u' + ku = 0 \\ u(0) = u_0 \\ u'(0) = u'_0 \end{cases}$ be the initial value problem for a spring.

Solve the initial value problem and notice that regarding the damping there are three cases:

1. If $mr^2 + \gamma r + k = 0$ has complex conjugate roots then system is **under-damped**.

   In this case solve the equation to find the **quasi-frequency** and refer to previous paragraph to describe the behavior.

2. If $mr^2 + \gamma r + k = 0$ has repeated roots solution then system is **critically damped**.

   Solve to find the motion. Refer to the previous paragraph for the behavior of the system.

3. If $mr^2 + \gamma r + k = 0$ has two distinct roots then system is **over-damped**.

   Solve to find the motion. Refer to the previous paragraph for the behavior of the system.

- **Damped forced motion**

  Solve as any nonhomogeneous second order.

Transient solution is the homogeneous part and damping make it vanish over time.

Steady state solution is the particular solution that does not vanish over time.

Resonance: When the forcing function has the same frequency as the natural frequency of the system:

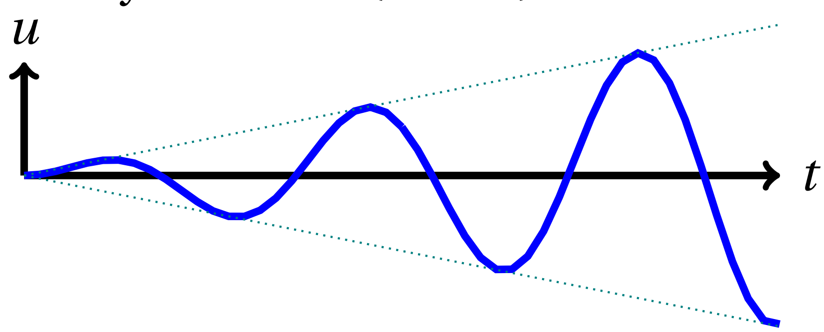
$mu'' + ku = F\cos(\omega_0 t)$ (similarly $mu'' + ku = F\sin(\omega_0 t)$)

Where $\omega_0^2 = \dfrac{k}{m}$

The solution to system starting at rest looks like:

$u = \frac{Ft}{2m\omega_0}\sin(\omega_0 t)$ Which is a resonance. Note that the solution is indefinitely increasing.

$y = .01\,t\cos(t - 0.3)$



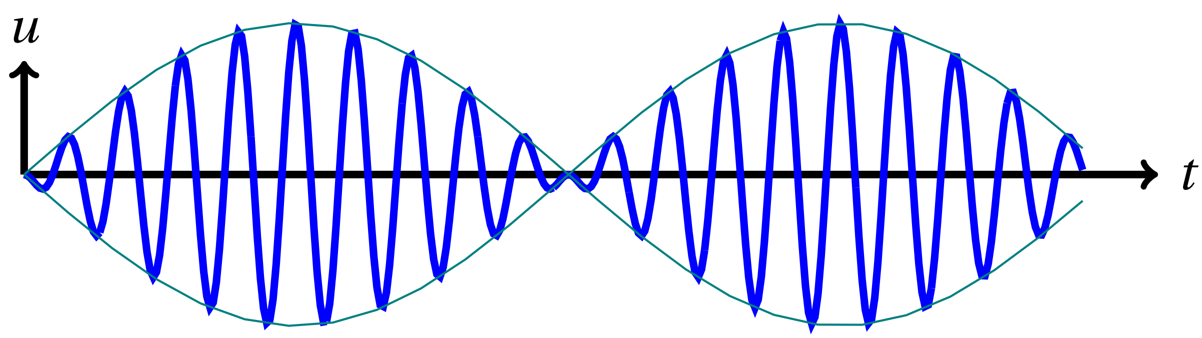
Beat $mu'' + ku = F\cos(\omega t)$. Where $\omega \neq \omega_0 = \sqrt{\frac{k}{m}}$ (the natural frequency of the system).

In this case the solution to the system starting at rest looks like:

$$u = \frac{1}{m(\omega_0^2 - \omega^2)}\Big(\cos(\omega t) - \cos(\omega_0 t)\Big) =$$

$$\frac{2}{m(\omega_0^2 - \omega^2)} sin\left(\frac{(\omega_0 - \omega)t}{2}\right) sin\left(\frac{(\omega_0 + \omega)t}{2}\right)$$

In this phenomena the amplitude changes periodically. This is used in am (**amplitude modulation**). It is only interesting when ω and ω_1 are“close" in value.


u
t

Example:

A 16 pound weight stretches a spring 2 feet. The medium through which the weight moves offers a resistance equal to 4 times the velocity in ft/sec. Find the position of the weight at any time t sec, if the weight is released with zero velocity from one foot **above** the equilibrium position. *

Solution:

Use the formula to find the spring constant of proportionality $k = \frac{16}{2} = 8$ lb/ft

$\gamma = 4$ and $m = \frac{16}{32} = .5$ poundal mass

$u(0) = 1$ ft and $u'(0) = 0$ ft/sec

*This is an initial position.

Thus

$$\begin{cases} .5u'' + 4u' + 8u = 0 \\ u(0) = 1 \;\; u'(0) = 0 \end{cases}$$

The characteristic equation has double root at $r = -4$

So the solution is $u(t) = C_1 e^{-4t} + C_2 t e^{-4t}$

$u' = -4C_1 e^{-4t} + C_2(e^{-4t} - 4te^{-4t})$

So the solution is $u(t) = e^{-4t} + 4te^{-4t}$.

Critically Damped

Example:

Repeat the above experiment with a medium that offers resistance 3 times the velocity in ft/sec. Find the quasi frequency and quasi period. What is the first time that the spring goes to equilibrium position? At what time does the amplitude of vibration become smaller that 0.01ft?

Solution:

$$\begin{cases} .5u'' + 3u' + 8u = 0 \\ u(0) = 1 \;\; u'(0) = 0 \end{cases}$$

The characteristic equation has double root at $r = -3 \pm \sqrt{9-16} = -3 \pm \sqrt{7}i$

Then the general solution is $u(t) = C_1 e^{-3t}\cos(\sqrt{7}t) + C_2 e^{-3t}\sin(\sqrt{7}t)$

$u(0) = C_1 = 1$ and $u' = C_1(-3e^{-3t}\cos(\sqrt{7}t) - \sqrt{7}e^{-3t}\sin(\sqrt{7}t)) + C_2(-3e^{-3t}\sin(\sqrt{7}t) + \sqrt{7}e^{-3t}\cos(\sqrt{7}t))$

$u'(0) = -3 + \sqrt{7}C_2 = 0$ so $C_2 = 3/\sqrt{7}$

$$u(t) = e^{-3t}\cos(\sqrt{7}t) + \frac{3}{\sqrt{7}}e^{-3t}\sin(\sqrt{7}t)$$

The quasi frequency is $\mu = \sqrt{7}$ and the quasi period is $\frac{2\pi}{\sqrt{7}}$

$$u(t) = \frac{4}{\sqrt{7}}e^{-3t}\cos(\sqrt{7}t - .848)$$

The spring mass is going back to equilibrium first at $t = \frac{\pi/2 + .848}{\sqrt{7}}$

Use your calculator. Graph $y = u(t)$, $y = 0.01$ and $y = -0.01$. Find the largest t-value where the curve intersect either of the lines. That is, for all $t > 1.65$ $|u(t)| < .01$