The variation of parameters method

Method of undetermined coefficients is simple but it does not support all functions.* As you have seen the derivatives have to have finitely many reappearing terms.

The variation of parameters works for all continuous function but integration and simplification can be very difficult.

Here is how it works:

Let $y'' + by' + cy = f(t)$ and the homogeneous solution is $y_h = C_1 y_1 + C_2 y_2$.

Then we assume that $y_p = v_1 y_1 + v_2 y_2$, where v_1 and v_2 are functions.

Find the derivatives of y_p:

$$y_p' = v_1' y_1 + v_2' y_2 + v_1 y_1' + v_2 y_2'$$

*It does support a more natural class of functions though.

[†] We can assume

$v_1' y_1 + v_2' y_2 = 0$.[‡]

Then $y_p' = v_1 y_1' + v_2 y_2'$ and $y_p'' = v_1' y_1' + v_2' y_2' + v_1 y_1'' + v_2 y_2''$

Now :

$y'' + p(t) y' + q(t) y =$ [§]

$$(v_1' y_1' + v_2' y_2' + v_1 y_1'' + v_2 y_2'') + p(t)(v_1 y_1' + v_2 y_2') + q(t)(v_1 y_1 + v_2 y_2) =$$
$$v_1\left(y_1'' + p(t) y_1' + q(t) y_1\right) + v_2\left(y_2'' + p(t) y_2' + q(t) y_2\right) + v_1' y_1' +$$
$$v_2' y_2' = v_1' y_1' + v_2' y_2' = f(t)$$

[†] It becomes clear later that this is possible because $\{y_1, y_2\}$ are linearly independent.

[‡] The original restriction for v_1 and v_2 gives non-unique answers, so we are able to put more restrictions without changing the outcome.

[§] Note that the coefficient of y'' is one.

In order to determine the functions v_1 and v_2, we need to solve v_1' and v_2' in the equations :

$$v_1' y_1 + v_2' y_2 = 0$$

$$v_1' y_1' + v_2' y_2' = f(t)$$

That is, v_1 can be found by the Cramer's rule method:

$$v_1' = \frac{\begin{vmatrix} 0 & y_2 \\ f(t) & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} \qquad \textit{and} \qquad v_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & f(t) \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}$$

We call the following the Wronskian of functions y_1 and y_2:

$$W(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

So $v_1' = \frac{-y_2 f(t)}{W(t)}$ and $v_2' = \frac{y_1 f(t)}{W(t)}$

That is

$$v_1 = \int \frac{-y_2 f(t)}{W(t)} dt \text{ and } v_2 = \int \frac{y_1 f(t)}{W(t)} dt$$

Now

$v_1 y_1 + v_2 y_2 = -y_1 \int \frac{y_2 f(t)}{W(t)} dt + y_2 \int \frac{y_1 f(t)}{W(t)} dt$ gives the particular solution.*

Note: The solution exists if and only if the Wronskian is not zero at least one point. Wronskian is not zero at least at one point if and only if the functions are linearly independent.

Another Note: We denote the Wronskian with $W(y_1(t), y_2(t))$ or $W(y_1, y_2)(t)$. This is to show that the Wronskian acts on function y_1, y_2 and is a function of t.

*This is unique up to a homogeneous solution.

Example:

Find the solution to the equation $y'' - y' - 2y = e^{3x}$.

Solution:

- The characteristic equation: $r^2 - r - 2 = 0$ to get $r = 2$ and $r = -1$

- $y_1 = e^{2x}$ and $y_2 = e^{-x}$

- $W(y_1, y_2) = -e^{2x}e^{-x} - 2e^{2x}e^{-x} = -3e^{x}$

- $$v_1 = \int \frac{e^{-x}e^{3x}}{3e^{x}}dx = \frac{e^{x}}{3}$$

  $$v_2 = -\int \frac{e^{2x}e^{3x}}{3e^{x}}dx = -\frac{e^{4x}}{12}$$

- $y_p = \frac{e^x}{3}(e^{2x}) - \frac{e^{4x}}{12}(e^{-x}) = \frac{e^{3x}}{4}$

  $y = C_1e^{2x} + C_2e^{-x} + \frac{e^{3x}}{4}$

Example:

Solve $y'' - 3y' - 18y = t$

Solution:

- The homogeneous solution is:

  $y_h = C_1 e^{6t} + C_2 e^{-3t}$

  The Wronskian is $W = -9e^{3t}$

  $$y_p = -e^{6t} \int \frac{e^{-3t} t}{-9e^{3t}} dt + e^{-3t} \int \frac{e^{6t} t}{-9e^{3t}} dt = -\frac{t}{18} + \frac{1}{108}$$

  The general solution is: $y = C_1 e^{6t} + C_2 e^{-3t} - \frac{t}{18} + \frac{1}{108}$

Example: $y'' + 4y' + 4y = t^2 e^{-2t}$

Solution:

- $y_h = C_1 e^{-2t} + C_2 t e^{-2t}$

- $y_1 = e^{-2t}$ and $y_2 = t e^{-2t}$

- $$W(e^{-2t}, te^{-2t}) = \begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & -2te^{-2t} + e^{-2t} \end{vmatrix} = e^{-4t}$$

- $v_1 = -\int \dfrac{(t^2 e^{-2t})(te^{-2t})}{e^{-4t}} dt = -\int t^3 dt = \dfrac{-t^4}{4} + C$

- $v_2 = \int \dfrac{(t^2 e^{-2t})(e^{-2t})}{e^{-4t}} dt = \int t^2 dt = \dfrac{t^3}{3} + C$

- $v_1 y_1 + v_2 y_2 = -\dfrac{t^4}{4} e^{-2t} + \dfrac{t^3}{3} te^{-2t} = \dfrac{1}{12} t^4 e^{-2t}$

- $y_p = \frac{1}{12} t^4 e^{-2t}$

- $\boxed{y = C_1 e^{-2t} + C_2 te^{-2t} + \dfrac{t^4}{12} e^{-2t}}$

- Note that this is an improvement over the method of undetermined coefficients.

Example:

$$y'' + 9y = 9\sec^2(3t), \qquad 0 \le t < \frac{\pi}{6}$$

Solution:

- $y_h = C_1 \cos(3t) + C_2 \sin(3t)$

- $y_1 = \cos(3t)$ and $y_2 = \sin(3t)$

- $W(\cos(3t), \sin(3t)) = \begin{vmatrix} \cos(3t) & \sin(3t) \\ -3\sin(3t) & 3\cos(3t) \end{vmatrix} = 3$

- $v_1 = -\int \frac{(9\sec^2(3t))(\sin(3t))}{3} dt = -\int \frac{\sin(3t)}{3\cos^2(3t)} dt =$

  $v_1 = -\frac{1}{\cos(3t)} + C$

- $v_2 = \int \frac{(9\sec^2(3t))(\cos(3t))}{3} dt = \int \frac{3}{\cos(3t)} dt =$

$v_2 = \ln|\tan 3t + \sec(3t)| + C$

- $v_1 y_1 + v_2 y_2 = -\frac{1}{\cos(3t)} \cos(3t) + \ln|\tan(3t) + \sec(3t)| \sin(3t) = y_p$

- $\boxed{y = C_1 \cos(3t) + C_2 \sin(3t) - 1 + \ln|\tan(3t) + \sec(3t)| \sin(3t)}$

Example: $y'' + y = t \sin(t)$

Solution:

- $y_h = C_1 \cos(t) + C_2 \sin(t)$

- $y_1 = \cos(t)$ and $y_2 = \sin(t)$

- $$W(\cos(t), \sin(t)) = \begin{vmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{vmatrix} = 1$$

- $$v_1 = -\int \frac{(t\sin(t))(\sin(t))}{1}dt = -\int t\sin^2(t)dt =$$

$$-\int t\Big(\frac{1-\cos(2t)}{2}\Big)dt = -\frac{t^2}{4} + \frac{t\sin(2t)}{4} + \frac{\cos(2t)}{8} + C$$

- $$v_2 = \int \frac{(t\sin(t))(\cos(t))}{1}dt = \int t^2 dt = \int \frac{t\sin(2t)}{2}dt =$$

$$-\frac{t\cos(2t)}{4} + \frac{t\sin(2t)}{8} + C$$

- $$v_1 y_1 + v_2 y_2 =$$

$$(-\frac{t^2}{4}+\frac{t\sin(2t)}{4}+\frac{\cos(2t)}{8})\cos t+(-\frac{t\cos(2t)}{4}+\frac{\sin 2t}{8})\sin(t) =$$

$$-\frac{t^2}{4}\cos(t)+\frac{t(\sin(2t)\cos t-\cos(2t)\sin(t))}{4}+\frac{(\cos(2t)\cos(t)+\sin(2t)\sin(t))}{4}$$

$$-\frac{t^2}{4}\cos(t) + \frac{t\sin(t)}{4} + \frac{\cos(t)}{4}$$

- Since $\frac{\cos(t)}{4}$ is part of homogeneous solution, $y_p =$

$$-\frac{t^2}{4}\cos(t) + \frac{t\sin(t)}{4}$$

- $$\boxed{y = C_1\cos(t) + C_2\sin(t) - \frac{t^2}{4}\cos(t) + \frac{t\sin(t)}{4}}$$

Example:

$$y'' + 2y' + 5y = e^{-t}\cos(2t)$$

Solution:

- $y_h = C_1 e^{-t}\cos(2t) + C_2 e^{-t}\sin(2t)$ and $W(t) = 2e^{-2t}$

- $v_1 = -\int \frac{\sin(2t)\cos(2t)}{2}\,dt = -\int \frac{sin(4t)}{4}dt = \frac{1}{16}\cos(4t)$

- $v_2 = \int \frac{\cos(2t)\cos(2t)}{2}\,dt = \int \frac{1+\cos(4t)}{4}\,dt =$
  $\frac{1}{4}(\frac{\sin(4t)}{4} + t) = \frac{1}{16}\sin(4t) + \frac{t}{4}$

- $y_p = \frac{1}{16}\Big(\cos(2t)\cos(4t) + \sin(2t)\sin(4t)\Big) + \frac{t}{4}e^{-t}\sin(2t)$
  $= \frac{1}{16}\sin(2t)e^{-t} + \quad \frac{t}{4}\sin(2t)e^{-t}$ (homo)

- $y = C_1 e^{-t}\cos(2t) + C_2 e^{-t}\sin(2t) + \frac{t}{4}e^{-t}\sin(2t)$

The variation of parameters method, Overview of the method

- Solve the homogeneous equation to find y_1 and y_2.

- Find the Wronskian:

$$W(y_1, y_2)(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

- Use these formulas to find: $\boxed{v_1 = -\int \frac{f(t)y_2}{W(t)}dt}$

$\boxed{v_2 = \int \frac{f(t)y_1}{W(t)}dt}$

- Write out $v_1y_1 + v_2y_2$ and omit any part that is a multiple of homogeneous solution if you want.*

*Particular solution is only unique up to the homogeneous solution.

- What is left of $v_1 y_1 + v_2 y_2$, after the process in the previous line, is the particular solution: y_p.

- The general solution is $y = C_1 y_1 + C_2 y_2 + y_p$