Applied differential Equations

Day 1: Intro and Notations

Definition:

Equations containing derivatives are **differential equations**.

Examples: 0.1. *1.* $y' = t^2 + 2$ *2.* $y'' + 4y' = \cos x$ *3.* $y''' + 2(y')^3 + \sin x = 4$

Terminology(Definitions)

- **Ordinary differential equation**: An ordinary differential equation (ode) is an equation between one independent variable, one dependent variable and the derivative(s) of the independent variable.

  Examples: 0.2. $y' = y^2 + t$ *is an ode.*

  An example of a partial differential equation is $u_{tt} = u_x$ *or Heat equation:* $u_t = cu_{xx}$

  Here the unknown dependent variable $u(x,t)$ *is a function of both* x *and* t*.*

- A solution to an ode is a function of the independent variable that can be plugged in for dependent variable in the equation.

  Examples: 0.3. *1.* $y = \frac{t^3}{3} + 2t + 5$ *is a solution for* $y' = t^2 + 2$ *because* $\underbrace{\left(\frac{t^3}{3} + 2t + 5\right)'}_{y'} = t^2 + 2.$

  2. $y = \sin(2x) + \frac{\cos(x)}{3}$ *is a solution for* $y'' + 4y = \cos x$*, because*

  $$\underbrace{\left(\sin(2x) + \frac{\cos(x)}{3}\right)''}_{y''} + \underbrace{4\left(\sin(2x) + \frac{\cos(x)}{3}\right)}_{4y} = \cos(x).$$

- **Order**: The order of an ode is the order of the **highest derivative** in the equation.

  Examples: 0.4. *In Example 0.1 : 1: is of order 1 and 2 is of order 2 and 3 is of order 3.*

- **Linear**: A linear ode is one in which the dependent variable and its derivatives appear linearly.

  Examples: 0.5. *1.* $y' = y \sin t + \ln t$ *is a linear ode:*

  This is a linear ode even though there are terms $sin(t)$ *and* $ln(t)$*. The independent variable t can appear nonlinearly in a linear ode.*

  2. $y' = t\sin(y)$ *is a non-linear ode because* y *is non-linear in* $\sin(y)$*.*

  The general form of **a linear ode** is $P_n(t)\frac{d^n}{dx^n}y + P_{n-1}(t)\frac{d^{n-1}}{dx^{n-1}}y + ... + P_0(t) = 0$

  The general form of a **1st-order linear ode** is

  $y' + p(t)y = g(t)$

- **Autonomous ode**: For an autonomous ode $f(t, y)$ is a function of y only. That is $y' = g(y)$.

  Example 0.6. $y' = \underbrace{y(1-y)}_{\text{Finction of y-only}}$ *is an autonomous ode.*

Definition:

The expression representing all possible solutions to the ode and is called the **general solution**. The geometrical representation of the general solution is a family of infinitely many curves called **integral curves**. Each curve is associated with one value of the integration constant.

General Form of a first-order ode

The general form of a first-order ordinary differential equation is

$y' = f(t, y)$

or

$y' = f(t, y), y(t_0) = y_0$

$y(t_0) = y_0$ is the initial condition which determines the value of integration constant.

Note:

Here t is the independent variable and $y(t)$ is the dependent variable. $\underbrace{\text{The goal is to determine the unknown function } y(t)}_{\text{The solution to the equation}}$ whose derivative satisfies the above condition and which passes through the point (t_0, y_0)

Example 0.7. *The solution to initial value problem* $y'(t) = t^2 + 2$, $y(3) = 5$ *is*

$$y(t) = y(3) + \int_3^t (s^2 + 2)ds = 5 + (s^3/3 + 2s)\Big|_3^t \qquad \boxed{y(t) = t^3/3 + 2t - 10}$$

Direction/Slope Fields for First-Order ode

Qualitative analysis is useful tool to verify numerical or analytic solutions. Even if an explicit formula is known, qualitative analysis is useful, since it can give a visual picture of the behavior of solutions to an ode.

One method of estimating the solutions to a differential equation is using the **Direction/slope fields**.

For a first order differential equation, we can calculate the slope of tangent line at any point (x, y) on the plane. To obtain a **direction field** choose various points on a grid then, at each point draw line segments with slopes of tangent line at that point.

Example 0.8. *Use slope fields to find the solution to equations $y' = y(y-3)$ at the following initial values:*

1. $y(1) = 4$
2. $y(1) = 3$
3. $y(1) = 1$
4. $y(1) = -1$

First calculate the slopes at few different (x, y) values: Hint: the critical values are $y = 0$ and $y = 3$ so include them.

(x, y)	$y' = y(y-3)$
$\vdots$	$\vdots$
$(0, -2)$	$y' = 10$
$(0, -1)$	$y' = 4$
$(0, 0)$	$y' = 0$
$(0, 1)$	$y' = -2$
$(0, 2)$	$y' = -2$
$(0, 3)$	$y' = 0$
$(0, 4)$	$y' = 4$
$\vdots$	$\vdots$

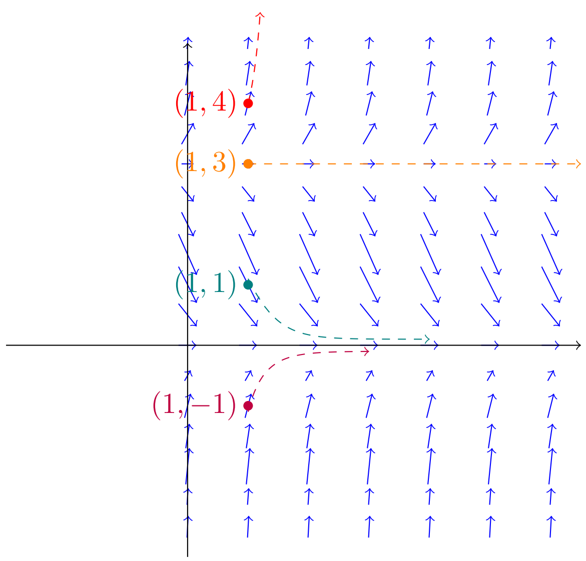
(x, y)	$y' = y(y-3)$
$\vdots$	$\vdots$
$(1, -2)$	$y' = 10$
$(1, -1)$	$y' = 4$
$(1, 0)$	$y' = 0$
$(1, 1)$	$y' = -2$
$(1, 2)$	$y' = -2$
$(1, 3)$	$y' = 0$
$(1, 4)$	$y' = 4$
$\vdots$	$\vdots$

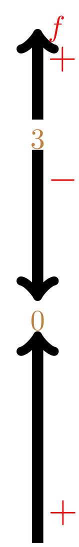
(x, y)	$y' = y(y-3)$
$\vdots$	$\vdots$
$(2, -2)$	$y' = 10$
$(2, -1)$	$y' = 4$
$(2, 0)$	$y' = 0$
$(2, 1)$	$y' = -2$
$(2, 2)$	$y' = -2$
$(2, 3)$	$y' = 0$
$(2, 4)$	$y' = 4$
$\vdots$	$\vdots$

(x, y)	$y' = y(y-3)$
$\vdots$	$\vdots$
$(3, -2)$	$y' = 10$
$(3, -1)$	$y' = 4$
$(3, 0)$	$y' = 0$
$(3, 1)$	$y' = -2$
$(3, 2)$	$y' = -2$
$(3, 3)$	$y' = 0$
$(3, 4)$	$y' = 4$
$\vdots$	$\vdots$

(x, y)	$y' = y(y-3)$
$\vdots$	$\vdots$
$(4, -2)$	$y' = 10$
$(4, -1)$	$y' = 4$
$(4, 0)$	$y' = 0$
$(4, 1)$	$y' = -2$
$(4, 2)$	$y' = -2$
$(4, 3)$	$y' = 0$
$(4, 4)$	$y' = 4$
$\vdots$	$\vdots$

Then graph a small line segment with those slopes at each (x, y);





Last start at each initial and see if you can find a solution. (The dashed lines are my approximations.) If you use more points for the slope field, you get a better approximation.

About Equilibrium Solutions:

1. *What are the equilibrium solutions?* $y = 0$ *and* $y = 3$ *where it seems that if the initial value is placed on either, the solution remain on that line.*
2. *Which of the equilibrium is a stable equilibrium?* $y = 0$*. The solutions seem to diverge from* $y = 3$ *but converge to* $y = 0$ *from surrounding initial points.*