Solving Homogeneous linear Systems

From the introduction, we can see that it is reasonable to assume that any homogeneous system has the same set of solutions. That is, we assume that the solution is an exponential function, where the exponent is either real or complex.

So we assume at least one of the solutions is exponential.

Let $\vec{x} = e^{\alpha t}\vec{\xi}$ where $\vec{\xi}$ is a constant vector, then

$\vec{x}' = \lambda e^{\alpha t}\vec{\xi}$ and $A\vec{x} = A\vec{\xi}e^{\alpha t}$

So $A\vec{\xi} = \lambda\vec{\xi}$ or $(A - \lambda I)\vec{\xi} = \vec{0}$ That is, λ is an eigenvalues and $\vec{\xi}$ is an eigenvectors of A.

When an eigenvalue λ is real: $\vec{\xi} e^{\lambda t}$ is one of fundamental solution of the equation, where $\vec{\xi}$ is the corresponding eigenvector to λ.

Example:

Find the general solution of the linear system:

$$\begin{cases} x_1' = 2x_1 - 3x_2 \\ x_2' = 4x_1 - 6x_2 \end{cases}$$

$$\vec{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 4 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

First find the eigenvalues:

Solve $\det\left(A-\lambda I\right)=0$

$$\det\left(\begin{bmatrix} 2-\lambda & -3 \\ 4 & -6-\lambda \end{bmatrix}\right)=\lambda^2+4\lambda=0$$

Implies $\lambda_1=0$ and $\lambda_2=-4$

For $\lambda_1=0$

$$\begin{bmatrix} 2 & -3 \\ 4 & -6 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix}=0$$

$rref$ the matrix and

$$\begin{bmatrix} 1 & -3/2 \\ 0 & 0 \end{bmatrix}$$

That is, there are infinite solutions with $k_1 - \frac{3}{2}k_2 = 0$.

$k_1 = \frac{3}{2}k_2$.

All solutions are of the form:

$$\begin{bmatrix} \frac{3}{2}k_2 \\ k_2 \end{bmatrix}$$

This is a vector subspace (A line through origin.) Choose one linearly independent vector:

$$\vec{\xi}_1 = \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}$$

we do the same for $\lambda_2 = -4$:

$$\begin{bmatrix} 6 & -3 \\ 4 & -2 \end{bmatrix}$$

rref gives:

$$\begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix}$$

That gives

$$\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} k_2$$

The fundamental set of solutions is

$$\left\{ \begin{bmatrix} 3/2 \\ 1 \end{bmatrix}, e^{-4t} \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} \right\}$$

$$\vec{x}(t) = C_1 \begin{bmatrix} 3/2 \\ 1 \end{bmatrix} + C_2 e^{-4t} \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{2}C_1 + \frac{1}{2}C_2 e^{-4t} \\ C_1 + C_2 e^{-4t} \end{bmatrix}$$

Find the ivp solution for the system if $\vec{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Plug in the initial values. Then solve for C_1 and C_2

$$\begin{cases} \frac{3}{2}C_1 + \frac{1}{2}C_2 = 1 \\ C_1 + C_2 = 0 \end{cases}$$

$C_1 = \dfrac{1}{2}$ and $C_2 = -\dfrac{1}{2}$

$$\vec{x}(t) = \frac{3}{2}\begin{bmatrix} 3/2 \\ 1 \end{bmatrix} + \frac{1}{2}e^{-4t}\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$$

Example: Do this in class:

$$\vec{x}' = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \vec{x}, \quad \vec{x}(0) = \begin{bmatrix} 0 \\ -4 \end{bmatrix}$$

Solution:

$\lambda = -1$ and $\lambda = 4$ gives $\vec{x}(t) = -\frac{8}{5} e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} - \frac{12}{5} e^{4t} \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}$

To check your answer, use this format: $\vec{x} = \begin{bmatrix} \frac{8}{5} e^{-t} - \frac{8}{5} e^{4t} \\ -\frac{8}{5} e^{-t} - \frac{12}{5} e^{4t} \end{bmatrix}$

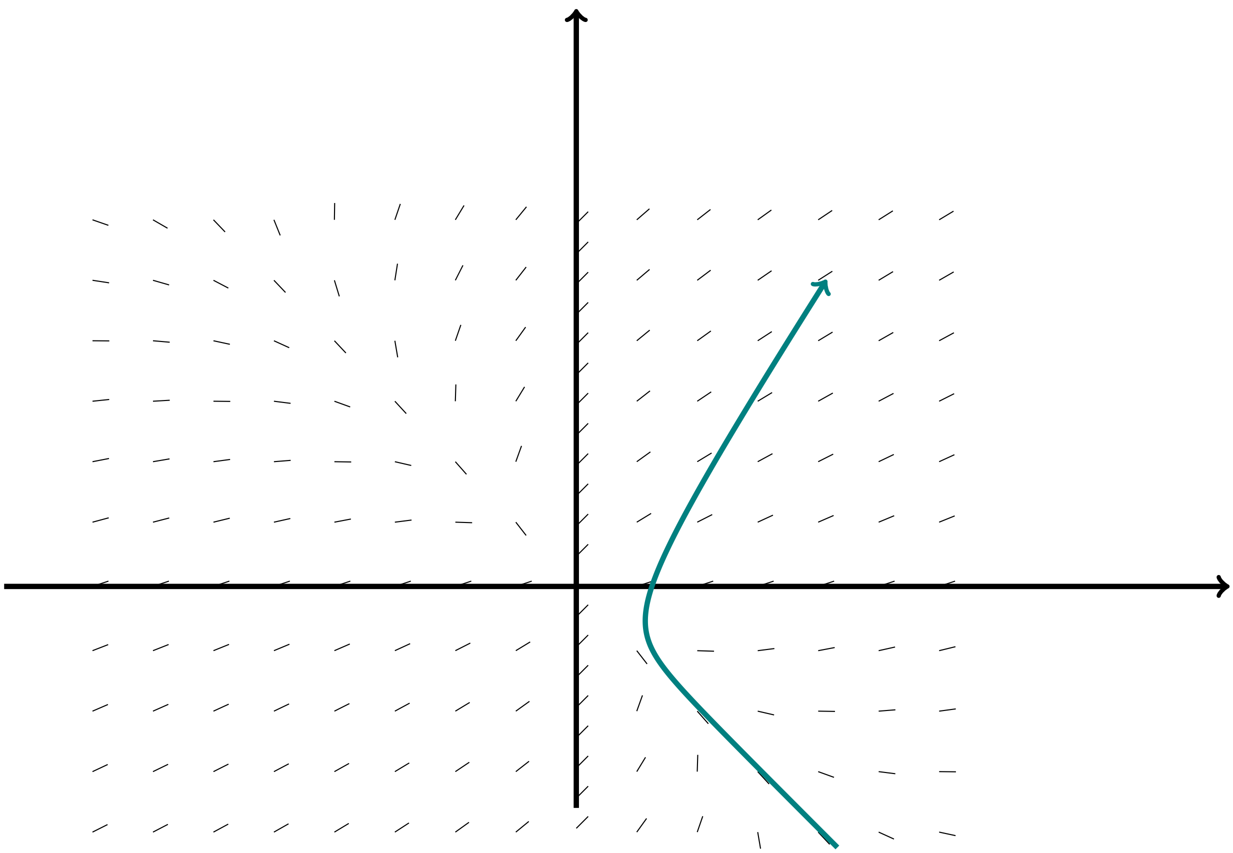
Phase planes

We can predict the behaviour of a two variables/ two equations system by finding a direction field with the slopes that are the ratio of the two derivatives. You can find the direction where the solution is going by this method. That is, as $t \to \infty$, the direction on the curve is an indication of where the two solutions are going.

Example: $\vec{x}' = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \vec{x} \qquad \vec{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Make the direction field by finding the vector $\vec{x}'$ at each point.

For example, at point $(2,3)$ the vector $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 8 \\ 12 \end{bmatrix}$ so the slope of change is $\frac{3}{2}$. Then use the slopes to find a curve through the initial value.



Eigenvalue of multiplicity 2 and eigenspace of dimension 2

When geometric multiplicity equal to algebraic multiplicity.

Find the general solution of the linear system:

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} 7 & 0 & -3 \\ -9 & -2 & 3 \\ 18 & 0 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Solution:

First find the eigenvalues:

Solve $\det\left(A - \lambda I\right) = 0$

$$\det\left(\begin{bmatrix} 7-\lambda & 0 & -3 \\ -9 & -2-\lambda & 3 \\ 18 & 0 & -8-\lambda \end{bmatrix}\right) = (\alpha+2)^2(\lambda-1) = 0$$

Implies a double eigenvalue $\lambda_1 = -2$ and a simple eigen-value $\lambda_2 = 1$

Find the corresponding vectors: $\lambda = -2$

$$\begin{bmatrix} 9 & 0 & -3 \\ -9 & 0 & 3 \\ 18 & 0 & -6 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = 0$$

$rref$ gives:

$$\begin{bmatrix} 1 & 0 & -1/3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore $k_1 - (1/3)k_3 = 0$. That is, $k_1 = 1/3k_3$

That gives

$$\begin{bmatrix} \frac{1}{3}k_1 \\ k_2 \\ k_1 \end{bmatrix} = k_1 \begin{bmatrix} \frac{1}{3} \\ 0 \\ 1 \end{bmatrix} + k_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Solution one is $\vec{x}_1(t) = e^{-2t} \begin{bmatrix} \frac{1}{3} \\ 0 \\ 1 \end{bmatrix}$

Solution two is $\vec{x}_2(t) = e^{-2t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

For a third linearly independent solution:

$$\begin{bmatrix} 6 & 0 & -3 \\ -9 & -3 & 3 \\ 18 & 0 & -9 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

rref:

$$\begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

which gives the eigenvector:

$$\begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix}$$

The general solution is :

$$\vec{x}(t) = C_1 \begin{bmatrix} \frac{1}{3} \\ 0 \\ 1 \end{bmatrix} e^{-2t} + C_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{-2t} + C_3 \begin{bmatrix} 0.5 \\ -0.5 \\ 1 \end{bmatrix} e^{t}$$

Eigenvalue of multiplicity bigger than the dimension of their eigenspace.

When algebraic multiplicity is bigger than geometric multiplicity. Let $\vec{\xi}$ be an eigenvector corresponding to repeated real root λ. One of the fundamental solutions will be $e^{\lambda t}\vec{\xi}$. We may be tempted to do the following:

Wrong assumption: Let's start with what we used to do with linear second order homogeneous odes. Assume $\vec{x} = te^{\lambda t}\vec{\xi}$ is another solution to $\vec{x}' = A\vec{x}$. Then plug it in:

$$e^{\lambda t}\vec{\xi} + \lambda t e^{\lambda t}\vec{\xi} = Ate^{\lambda t}\vec{\xi}$$

Then $e^{\lambda t}\vec{\xi} = \vec{0}$, a contradiction.

So, we need to add a term in the right hand side of the equation to balance the equation.

Let $\vec{x} = te^{\lambda t}\vec{\xi} + e^{\alpha t}\vec{\eta}$

Plug in the system:

$$e^{\lambda t}\vec{\xi} + \lambda t e^{\lambda t}\vec{\xi} + \lambda e^{\lambda t}\vec{\eta} = A(te^{\lambda t}\vec{\xi} + e^{\lambda t}\vec{\eta})$$

That is, if $\vec{\xi}$ is an eigenvector, then $(A - \lambda I)\vec{\eta} = \vec{\xi}$.

Process of finding solution:

Step 1: Find the eigenvector $\vec{\xi}$ associated with λ.

Step 2: Find $\vec{\eta}$ such that $(A - \lambda I)\vec{\eta} = \vec{\xi}$ (η is called **Generalized Eigenvector**)

Step 3: The solution is $\vec{x}(t) = C_1\vec{\xi}e^{\alpha t} + C_2\left(\vec{\xi}te^{\alpha t} + \vec{\eta}e^{\alpha t}\right)$

Example:

Find the general solution for the following system.

$$\begin{cases} x_1' = x_1 - 4x_2 \\ x_2' = 4x_1 - 7x_2 \end{cases} \qquad \vec{x}(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ 4 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Find one of the eigenvectors:

$$\begin{bmatrix} 4 & -4 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$k_1 = k_2$ gives the eigenvector:

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The first solution to the system is:

$$\vec{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-3t}$$

To find **the generalized eigenvector**, solve the following system.

$$\begin{bmatrix} 4 & -4 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

That is, $4k_1 = 4k_2 + 1$ so all solutions are $\begin{bmatrix} k_2 + \frac{1}{4} \\ k_2 \end{bmatrix}$. One generalized eigenvector is $\begin{bmatrix} \frac{1}{4} \\ 0 \end{bmatrix}$.

$$\vec{x}_2(t) = te^{-3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{-3t} \begin{bmatrix} 1/4 \\ 0 \end{bmatrix}$$

Dropping the last term, the general solution is:

$$\vec{x}(t) = C_1 e^{-3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2\left(e^{-3t}\begin{bmatrix} 1/4 \\ 0 \end{bmatrix} + te^{-3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$$

The initial value solution is :

$$\vec{x}(t) = e^{-3t}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 4te^{-3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Another example for you to do in class:

$$\vec{x}\,' = \begin{bmatrix} 7 & 1 \\ -4 & 3 \end{bmatrix}\vec{x} \qquad \vec{x}(0) = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$$

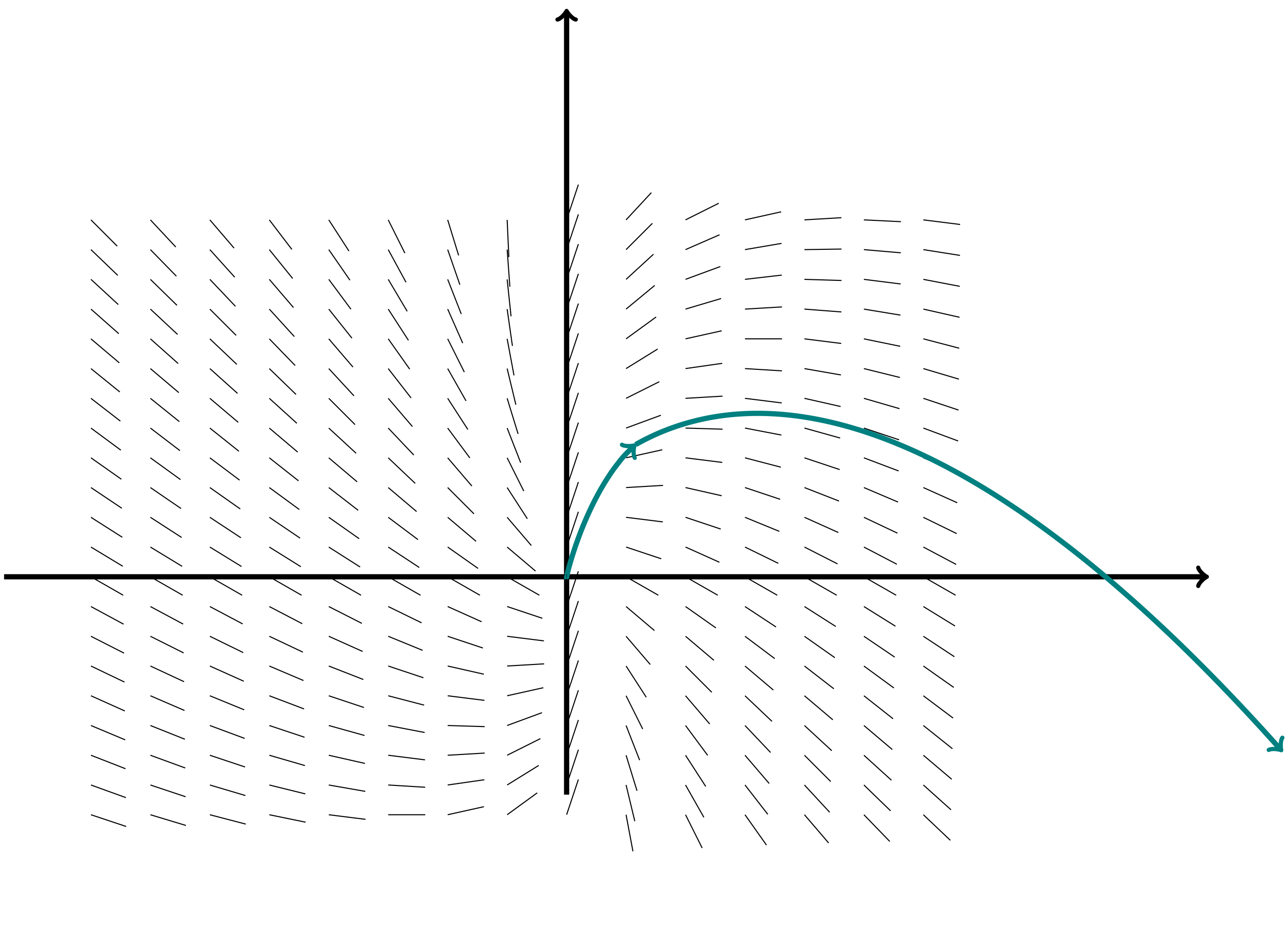
Solution: $\det(A - \lambda I) = \vec{0}$ implies $\lambda_1 = \lambda_2 = 5$

$$\vec{x}(t) = C_1 e^{5t}\begin{bmatrix} -0.5 \\ 1 \end{bmatrix} + C_2\left(te^{5t}\begin{bmatrix} -0.5 \\ 1 \end{bmatrix} + e^{5t}\begin{bmatrix} -0.5 \\ 0 \end{bmatrix}\right)$$

The initial value solution is: $\vec{x}(t) = e^{5t}\begin{bmatrix} 2 \\ -5 \end{bmatrix} - te^{5t}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

Another Phase Plane: Repeated root.

$$\vec{x}' = \begin{bmatrix} 7 & 1 \\ -4 & 3 \end{bmatrix} \vec{x} \qquad \vec{x}(0) = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$$



Real 2×2 matrices with complex eigenvalues:

Step 1: Find the eigenvalues and form the matrix

$A - \lambda I = A - (\alpha + i\omega)I$.

Step 2: If A is 2×2, then use the row that is more convenient, psst the second row, to calculate the eigenvector $\vec{\xi}$ associated with $\lambda_1 = \alpha + i\omega$. *

Step 3: If A had all real entries, then $\lambda_2 = \overline{\lambda_1}$ is an eigenvalue of A and the complex conjugate of the first eigenvector is an eigenvector corresponding to the second eigenvalue. That is, $\overline{\vec{\xi}}$ is an eigenvector corresponding to $\overline{\lambda}$.

Step 4: Then we have:

$$\vec{x}(t) = C_1 \vec{\xi} e^{\alpha t}\Big(\cos(\omega t) + i \sin(\omega t)\Big) + C_2 \overline{\vec{\xi}} e^{\alpha t}\Big(\cos(\omega t) - i \sin(\omega t)\Big)$$

*Remember that since $A - \lambda I$ is singular, the two rows must be linearly dependent.

In practice, we want to find the real solution and not the complex solution.

Step 5: Compute the real part R and the imaginary part G of $\vec{\xi}\Big(\cos(\omega t) + i\sin(\omega t)\Big)$

Step 6: $C_1 e^{\alpha t} R + C_2 e^{\alpha t} G$ is the general solution.

That is,

$$\vec{x} = C_1\Big(\Re(\vec{\xi})\cos(\omega t) - \Im(\vec{\xi})\sin(\omega t)\Big) + C_2\Big((\Im(\vec{\xi})\cos(\omega t) + \Re(\vec{\xi})\sin(\omega t)\Big)$$

Example:

Find the general solution of the linear system:

$$\begin{cases} x_1' = x_1 - 5x_2 \\ x_2' = x_1 - 3x_2 \end{cases} \qquad \vec{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Solution:

The characteristic equation: $\lambda^2 + 2\alpha + 2 = 0$

$\lambda = -1 \pm i$

$A - (-1+i)I = \begin{bmatrix} 2-i & -5 \\ 1 & -2-i \end{bmatrix}$ and

the corresponding eigenvectors $v = \begin{bmatrix} 2+i \\ 1 \end{bmatrix}$

So the general solution is

$$\vec{x}(t) = C_1 e^{-t}\begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + C_2 e^{-t}\begin{pmatrix} 2\sin t + \cos t \\ \sin t \end{pmatrix}$$

Now find the initial value problem:

$$\vec{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Plug in the initial values. Then solve for C_1 and C_2

$$\begin{cases} 2C_1 + C_2 = 1 \\ C_1 = 0 \end{cases}$$

$C_1 = 0$ and $C_2 = 1$

Another example for you to solve in class;

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 3 & -9 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Solution:

$\lambda = \pm 3i$

For $\lambda_1 = 3i$

$$\begin{bmatrix} 3-3i & -9 \\ 2 & -3-3i \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So eigenevtor

$$\begin{bmatrix} 3+3i \\ 2 \end{bmatrix}$$

$$\vec{x}(t) = C_1 \begin{bmatrix} 3\cos 3t - 3\sin 3t \\ 2\cos 3t \end{bmatrix} + C_2 \begin{bmatrix} 3\sin 3t + 3\cos 3t \\ 2\sin 3t \end{bmatrix}$$

The following is the solution to a purely imaginary eigenvalue.