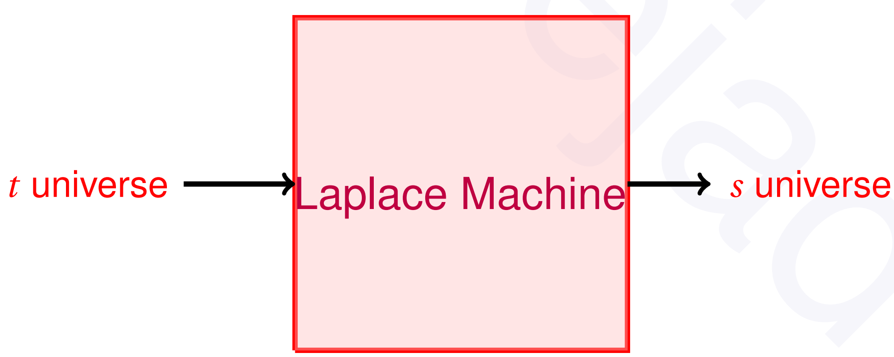
Introduction to Laplace transform

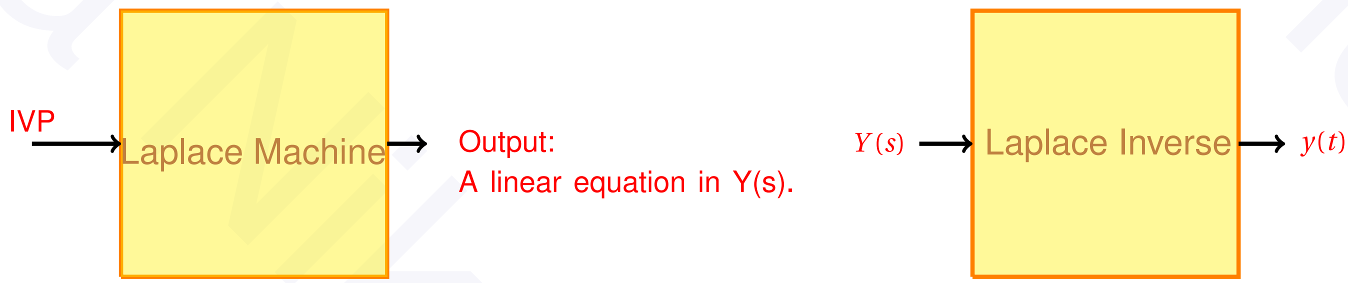
Some Scattered Notes:

- An integral transform. (You will see)
- It sums up the entire behaviour of a function starting at time $t = 0$ to infinity.
- It gives back an output that is somewhat equivalent to dissecting the function into different frequencies.
- Fourier transform is another integral transform that actually dissects into different frequencies.
- So in a way, Laplace transform vanishes time and signifies other properties of the function.



How Does it Work for IVPs?

Laplace operator transforms an ODE in $y(t)$ into a linear equation of $Y(s) = \mathscr{L}\big(y(t)\big)$ without any derivatives. The after solving for $Y(s)$ in a linear equation in the s-universe, we take Laplace inverse to find $y(t)$. *



Other than transforming a differential equation into a linear equation, the other significance of Laplace transform is that it facilitate working with certain discontinuities . Namely, it makes working with Heaviside and impulse function much easier.

*We denote $Y(s) = \mathscr{L}\big(y(t)\big)$. Use capital letters in s universe and lower case in t universe. $\mathscr{L}$ denotes Laplace transform. $\mathscr{L}^{-1}$ denotes the inverse of Laplace transform.

The Laplace Transform of a function $y(t)$ is defined by:

$$Y(s) = \mathscr{L}[y(t)](s) = \int_0^\infty e^{-st} y(t) dt$$

if the integral exists. $\mathscr{L}[y(t)](s)$ denotes the Laplace transform of $y(t)$. The functions $y(t)$ and $Y(s)$ are partner functions. Note that $Y(s)$ is a function of s only since the definite integral is with respect to t.

Note that f has to be of exponential order a, That is, $|f(t)| \leq ke^{at}$ for some constant a, to make the improper integral convergent.

Examples: (We are building a table out of these examples.)

- Find $\mathscr{L}\left(1\right)$

$$\mathscr{L}(1) = \int_0^\infty e^{-st}\,dt = \lim_{b\to\infty} -\frac{e^{-st}}{s}\Big|_0^b = \frac{1}{s}$$

- Find $\mathscr{L}\left(e^t\right)$

Let $y(t) = e^t$, then $Y(s) = \int_0^\infty e^{-st}e^t\,dt = \int e^{-(s-1)t}\,dt = \frac{1}{s-1}$

The integral converges if $s > 1$. The functions e^t and $\frac{1}{(s-1)}$ are partner functions.

- $\mathscr{L}\left(e^{at}\right) = \frac{1}{s-a}$ for $s > a$.

This is similar to the previous statement.

• Find $\mathscr{L}\Big(\cos(3t)\Big)$.

Let $y(t) = \cos(3t)$. We have

Method 1: Integration by parts twice:

$$\int_0^\infty e^{-st}\cos(3t)\,dt = \lim_{b\to\infty}\frac{1}{3}e^{-st}\sin(3t)\Big|_0^b - \lim_{b\to\infty}\frac{s}{9}e^{-st}\cos(3t)\Big|_0^b - \frac{s^2}{9}\int_0^\infty e^{-st}\cos(3t)\,dt$$

Replace $\int_0^\infty e^{-st}\cos(3t)\,dt$ by $\mathbf{I}$ and use squeeze theorem:

$$\mathbf{I} = \underbrace{\lim_{b\to\infty}\frac{1}{3}e^{-st}\sin(3t)\Big|_{t=0}^{t=b}}_{\to 0} - \underbrace{\lim_{b\to\infty}\frac{s}{9}e^{-st}\cos(3t)\Big|_0^b}_{\to \frac{s}{9}} - \frac{s^2}{9}\mathbf{I}$$

then solve for $\mathbf{I}$ to get: $\mathbf{I} = \dfrac{s}{s^2+9}$. So

$$Y(s) = \mathscr{L}\Big(\cos(3t)\Big) = \int_0^\infty e^{-st}\cos(3t)\,dt = \frac{s}{s^2+9}$$

Convergent for $s > 0$.

Method 2:

By Euler identity $\quad \text{real}\left(e^{-st+3ti}\right) = e^{-st}\cos(3t)$.

$$\int_0^\infty e^{-st}\cos(3t)\ dt = \text{real}\left(\int_0^\infty e^{-st+3ti}\ dt\right) =$$

$$\text{real}\left(\lim_{b\to\infty} \frac{e^{-st+3it}}{-s+3i}\Big|_{t=0}^{t=b}\right) = \lim_{b\to\infty} \text{real}\left(\frac{e^{-st+3ti}}{-s+3i}\Big|_{t=0}^{t=b}\right) =$$

$$\lim_{b\to\infty} \text{real}\left(\frac{e^{-sb+3bi}-1}{-s+3i}\right) =$$

$$\lim_{b\to\infty} \text{real}\left(\frac{(-s-3i)(e^{-sb}\cos(3b)+ie^{-sb}\sin(3b)-1)}{s^2+9}\right) =$$

$$\lim_{b\to\infty} \frac{-se^{-sb}\cos(3b)+3e^{-sb}\sin(3b)+s}{s^2+9} = \frac{s}{s^2+9}$$

So $\mathscr{L}\left(\cos(3t)\right) = \dfrac{s}{s^2+9}$

- Find $\mathscr{L}\left(t\cos(\alpha t)\right)$.

Now let $y(t) = t\cos(\alpha t)$

$$\mathscr{L}\left(t\cos(\alpha t)\right) = \int_0^\infty te^{-st}\cos(\alpha t)dt$$

Let's start with $\mathscr{L}\left(\cos(\alpha t)\right) = \int_0^\infty e^{-st}\cos(\alpha t)dt$

Take the derivative with respect to s and use Leibniz Rule of integration*

$$\frac{\partial}{\partial s}\mathscr{L}\left(\cos(\alpha t)\right) = \frac{\partial}{\partial s}\left(\int_0^\infty e^{-st}\cos(\alpha t)\right)dt = \int_0^\infty \frac{\partial}{\partial s}\left(e^{-st}\cos(\alpha t)dt\right)$$

$$= -\int_0^\infty te^{-st}\cos(\alpha t)dt = \mathscr{L}\left(t\cos(\alpha t)\right)$$

So $\mathscr{L}\left(t\cos(\alpha t)\right) = \frac{\partial}{\partial s}\mathscr{L}\left(\cos(\alpha t)\right) = -\frac{\partial}{\partial s}\left(\frac{s}{s^2+\alpha^2}\right) = -\frac{a^2-s^2}{(s^2+a^2)^2}$

So $\mathscr{L}\left(t\cos(\alpha t)\right) = \frac{s^2-a^2}{(s^2+a^2)^2}$

*Next slide explains the Leibniz rule of integration.

Theorem If $f(t,s)$ is a function of two variable such that $f(t,s)$ and $f_s(t,s)$ are continuous on $a(s) < t < b(s)$ for s in an open interval. If $a(s)$ and $b(s)$ are both differentiable over that interval. Then

$$\frac{\partial}{\partial s}\int_{a(s)}^{b(s)} f(t,s)dt = \int_{a(s)}^{b(s)} \frac{\partial f(t,s)}{\partial s}dt + f(b(s),s)b'(x) - f(a(s),s)a'(s)$$

The theorem is discussing a more general form that we needed. Specifically, if a and b are constants, then the following hold and is called Leibniz Rule of integration:

$$\frac{\partial}{\partial s}\int_{a}^{b} f(t,s)dt = \int_{a}^{b} \frac{\partial f(t,s)}{\partial s}dt.$$

A Linear Operator

Remember that Laplace transform is an integral operator so it is a linear operator:

- $\mathscr{L}(f_1(t)+f_2(t)) = \mathscr{L}(f_1(t)) + \mathscr{L}(f_2(t))$.
- $\mathscr{L}(rf(t)) = r\mathscr{L}(f(t))$ where r is a constant.

Examples

- $\mathscr{L}(3) = 3\mathscr{L}(1) = \frac{3}{s}$.
- $\mathscr{L}(e^{-2t}+5e^{-3t}) = \frac{1}{s+2} + \frac{5}{s+3}$
- $\mathscr{L}(\frac{1}{2}cos(3t)) = \frac{s}{2(s^2+9)}$

- $\mathscr{L}(\frac{1}{2}sin(3t)) = \frac{3}{2(s^2+9)}$

- $\mathscr{L}(\frac{1}{2}e^{2t}cos(3t)) = \frac{s-2}{2((s-2)^2+9)}$

Step Functions and Heaviside

Let

$$f(t) = \begin{cases} 1 & 0 \le t < 1 \\ k & t = 1 \\ 0 & 1 < t \end{cases}$$

$$\mathcal{L}(f(t)) = \int_0^\infty f(t)e^{-st}\,dt = \int_0^1 e^{-st}\,dt = \frac{-e^{-st}}{s}\bigg|_0^1 = \frac{1-e^{-s}}{s}$$

Heaviside is defined:

Let

$$u_c(t) = \begin{cases} 0 & t < c \\ 1 & c \le t \end{cases}$$

- Find $\mathcal{L}\left(u_C(t)\right)$.

$$\mathcal{L}(u_c(t)) = \int_0^\infty u_c(t)e^{-st}\,dt = \int_c^\infty e^{-st}\,dt = \lim_{b\to\infty} \frac{-e^{-st}}{s}\bigg|_c^b = \frac{e^{-sc}}{s}$$

- Find $\mathscr{L}(u_c(t)f(t-c))$ for any $f(t)$ of "good" order type.

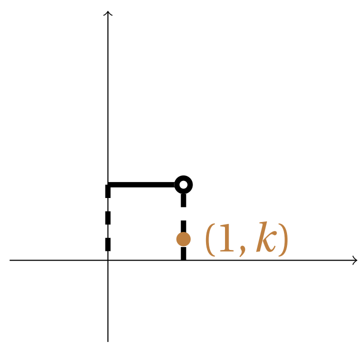
$$\mathscr{L}(u_c(t)f(t-c)) = \int_0^\infty u_c(t)e^{-st}f(t-c)\,dt =$$

$$\int_c^\infty e^{-st}f(t-c)\,dt = \int_0^\infty e^{-su}e^{-sc}f(u)\,du = e^{-sc}F(u) =$$

$e^{-sc}F(s)$.

Using u-substitute $u = t - c$.

Comparing an step function with Heaviside:



$f(t) = u_0(t) - u_1(t)$ for $k = 1$ and hence the steps can be made using the u functions. While finding Laplace transform, we can ignore a value of a function at one point. So the value k does not matter.

That is, **Heaviside** in engineering can be defined to be:

$$u_c(t) = \begin{cases} 0 & t < 0 \\ k & c = t \\ 1 & c < t \end{cases}$$

Piecewise Continuous : Domain of f can be partition into finite intervals where f is continuous and at points of discontinuity the limit is defined (the limits is not to infinity.)

Some examples of **piecewise defined** functions that can be built using the Heaviside steps:

$$f(t) = \begin{cases} 0 & t < 0 \\ 2 & 0 \le t < 3 \\ t-1 & 3 \le t < 5 \\ 0 & 5 \le t \end{cases}$$

$$f(t) = 2u_0(t) - 3u_3(t) + tu_3(t) - tu_5(t) + u_5(t)$$

Check: $2u_0(t) + (t-3)u_3(t) + (-t+1)u_5(t) =$

$$\begin{cases} 0 & t < 0 \\ 2 & 0 \le t < 3 \\ 2 & 3 \le t < 5 \\ 2 & 5 \le t \end{cases} + \begin{cases} 0 & t < 0 \\ 0 & 0 \le t < 3 \\ t-3 & 3 \le t < 5 \\ t-3 & 5 \le t \end{cases} + \begin{cases} 0 & t < 0 \\ 0 & 0 \le t < 3 \\ 0 & 3 \le t < 5 \\ -t+1 & 5 \le t \end{cases}$$

Example:

- Find Laplace transform of

$2u_0(t) + (t-3)u_3(t) + (-t+1)u_5(t)$

$$\mathscr{L}\left(2u_0(t) + (t-3)u_3(t) + (-t+1)u_5(t)\right) =$$

$$\mathscr{L}\left(2u_0(t) + (t-3)u_3(t) + (-t+5-4)u_5(t)\right) =$$

$$\frac{2}{s} + \frac{e^{-3t}}{s^2} + \frac{-e^{-5t}}{s^2} - \frac{4e^{-5t}}{s}$$

Existence of the Laplace Transform

If $y(t)$ is piecewise continuous for $t \geq 0$ and of exponential order, then the Laplace Transform exists for some values of s. A function $y(t)$ is of exponential order r if there is exist constants M and T such that

$|y(t)| \leq Me^{rt}$ where $t \geq T$

All polynomials, linear exponentials e^{at}, where a is a constant), sine and cosine functions, and products of these functions are of exponential order.

An example of a function not of constant exponential order is e^{t^2}. This function grows too rapidly. The integral

$$\int_0^{\infty} e^{-st} e^{t^2} \, dt$$

does not converge for any value of s.

The following table lists the Laplace Transforms for a selection of common functions.

Table of important Laplace transforms

$f(t)$	$\mathscr{L}(f(t)) = F(s)$	
1	$\frac{1}{s}$	$s > 0$
$e^{at} f(t)$	$F(s-a)$	$s > a$
e^{at}	$\frac{1}{(s-a)}$	$s > a$
te^{at}	$\frac{1}{(s-a)^2}$	$s > a$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$	$s > a$
$e^{at} \sin(bt)$	$\frac{b}{(s-a)^2 + b^2}$	$s > a$
$e^{at} \cos(bt)$	$\frac{s-a}{(s-a)^2 + b^2}$	$s > a$

Table of important Laplace transforms

$f(t)$	$\mathscr{L}(f(t)) = F(s)$	
$u_c(t)$	$\dfrac{e^{-cs}}{s}$	$s > 0$
$f(t-c)u_c(t)$	$e^{-cs}F(s)$	
$\delta(t)$	1	$s > 0$
$\delta(t-t_0)$	e^{-st_0}	$s > 0$
$\dfrac{df(t)}{dt}$	$sF(s) - f(0)$	
$\dfrac{d^n f(t)}{dt^n}$	$s^nF(s) - s^{n-1}f(0) - \ldots - f^{(n-1)}(0)$	

Table of important Laplace transforms

$f(t)$	$\mathscr{L}(f(t)) = F(s)$
$e^{at}\sinh(bt)$	$\dfrac{b}{(s-a)^2 - b^2}$
$e^{at}\cosh(bt)$	$\dfrac{s-a}{(s-a)^2 - b^2}$
$t\sin(bt)$	$\dfrac{2bs}{(s^2+b^2)^2}$
$t\cos(bt)$	$\dfrac{s^2-b^2}{(s^2+b^2)^2}$
$t\sinh(bt)$	$\dfrac{2bs}{(s^2-b^2)^2}$
$t\cosh(bt)$	$\dfrac{s^2-b^2}{(s^2-b^2)^2}$

- Find $\mathscr{L}(e^{at}f(t)) = F(s-a)$ for any $f(t)$ of order type r.

$$\int_0^\infty e^{(-s+a)t} f(t)\,dt = \int_0^\infty e^{-ut} f(t)\,dt = F(u) = F(s-a)$$

Using $u = s - a$.

- Find $\mathscr{L}(y'(t))$, when $y(0) = y_0$

$$\int_0^\infty e^{-st} y'(t)dt = \lim_{b\to\infty} e^{-st}y(t)\Big|_0^b - \int_0^\infty (-s)e^{-st}y(t)dt =$$

$$y(0) + s\mathscr{L}(y(t)).$$

Using integration by parts where $u = e^{-st}$ and $dv = y'(t)dt$.

So $\mathscr{L}(y'(t)) = s\mathscr{L}(y(t)) + y(0) = sY(s) + y(0)$

Similarly, $\mathscr{L}(y^{(n)}(t)) = s^nY(s) - s^{n-1}y(0) - ... - y^{(n-1)}(0)$.

Using n integration by parts.

Example

Find the Laplace transform of $y'' + y' - 6y = 17e^{-2t}\cos(3t)$, $y(0) = 5,\ y(0) = 7$ and solve for $Y(s)$.

$$s^2Y(s) - sy(0) - y'(0) + sY(s) - y(0) - 6Y(s) = \frac{17(s+2)}{(s+2)^2+9}$$

$$s^2Y(s) - 5s - 7 + sY(s) - 5 - 6Y(s) = \frac{17(s+2)}{(s+2)^2+9}$$

$$Y(s) = \frac{5s+12}{(s+2)^2+9} + \frac{17(s+2)}{[(s+2)^2+9][s^2+s-6]}$$

- The convolution integral:

$$F(s)G(s) = \int_0^{\infty} e^{-s\xi} f(\xi)\, d\xi \int_0^{\infty} e^{-s\tau} g(\tau)\, d\tau$$

$$= \int_0^{\infty} f(\xi) \int_0^{\infty} e^{-s(\xi+\tau)} g(\tau)\, d\tau\, d\xi =$$

Change of the variable for $\tau = t - \xi$ where ξ is fixed:

$$\int_0^{\infty} f(\xi) \int_{\xi}^{\infty} e^{-st} g(t-\xi)\, dt\, d\xi$$

Change the order of the integration:

$$\int_0^{\infty} e^{-st} \int_0^{t} g(t-\xi) f(\xi)\, d\xi\, dt$$

So $F(s)G(s) = \int_0^{\infty} e^{-st} \int_0^{t} g(t-\xi) f(\xi)\, d\xi\, dt$

- Dirac's delta. The **unit impulse function** which is zero everywhere except at time 0 which is large enough to have impart a pulse of size 1. (Impulse is the integral of force over time or the rate of change in momentum or if the forcing function is a current, the impulse represents voltage.

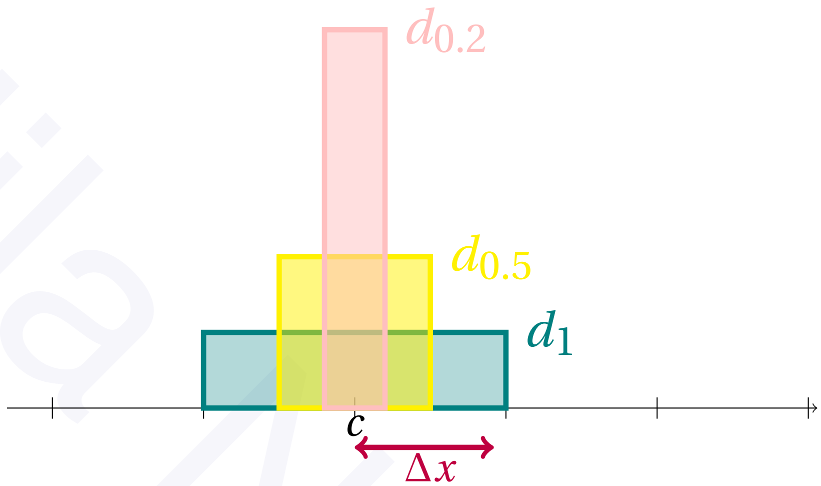
$$d_\Delta(t-c) = \begin{cases} \frac{1}{2\Delta} & t \in (c-\Delta, c+\Delta) \\ 0 & t \notin (c-\Delta, c+\Delta) \end{cases}$$

$$\mathscr{L}(\delta(t-c)) = \lim_{\Delta \to 0} \int_0^\infty e^{-st} d_\Delta(t-c)\,dt =$$

$$\lim_{\Delta \to 0} \int_{c-\Delta}^{c+\Delta} \frac{1}{2\Delta} e^{-st}\,dt = \lim_{\Delta \to 0} \frac{e^{-st}}{-2s\Delta}\Big|_{c-\Delta}^{c+\Delta} =$$

$$e^{-sc} \lim_{\Delta \to 0} \frac{(e^{-s\Delta} - e^{-2s\Delta})}{-2s\Delta} =$$

Use L'hospital to get : $\boxed{e^{-sc}}$



Each area is equal to one.

Note that impulse is infinite but the area under integral is limited.

$$\int_0^{\infty} \delta(t-c)\,dt = 1$$