Laplace Inverse

Uniqueness

the Laplace operator is not a one-to-one function. So to be able to define the inverse, we need to have an extra condition. We use the table to enforce a uniqueness condition for inverse.[*]

Linear Operator

With this definition, Laplace inverse is a linear operator. That is, if $\mathscr{L}^{-1}\Big(Y_1(s)\Big)(t) = y_1(t)$, $\mathscr{L}^{-1}\Big(Y_2(s)\Big)(t) = y_2(t)$ and c_1 and c_2 are constants, then

$$\mathscr{L}^{-1}\Big[c_1 Y_1(s)+c_2 Y_2(s)\Big]](t) = c_1\mathscr{L}^{-1}\Big(Y_1(s)\Big)(t)+c_2\mathscr{L}^{-1}\Big(Y_2(s))\Big)(t) = c_1 y_1(t) + c_2 y_2(t).$$[†]

[*] For example, if two functions are different only in finitely many points, then their Laplace is the same.

[†] The t with the Laplace inverse symbol shows that the inverse is with respect to t. Since in this course we always finding inverse with respect to t, we are going to omit it from now on.

Examples

- What is the inverse transform of $\frac{7}{s^3}$?

Use the table: $\mathscr{L}^{-1}\left(2/s^3\right) = y(t) = t^2$.

So $\mathscr{L}^{-1}\left(\frac{7}{s^3}\right) = \frac{7}{2}t^2$

- Find the Laplace inverse of $Y(s) = \dfrac{3s+4}{s^2+4}$.

According to the table, $\mathscr{L}^{-1}\left(\dfrac{s}{(s^2+4)}\right) = \cos(2t)$

and $\mathscr{L}^{-1}\left(\dfrac{2}{(s^2+4)}\right) = \sin(2t)$. So

$$\mathscr{L}^{-1}\left(\frac{3s+4}{s^2+4}\right) = 3\mathscr{L}^{-1}\left(\frac{s}{s^2+4}\right) + 2\mathscr{L}^{-1}\left(\frac{2}{s^2+4}\right) =$$

$3\cos 2t + 2\sin 2t$

In all these examples, the important part is the denominator. So

- Match the format of the denominators to the table.
- factor th denominators into linear factors and quadratic factors and their powers.
- If the format of the denominator is not appearing in the table, use partial fraction decomposition to find the format on the table.
- Match the format of numerators to the table.

Recall partial fraction decomposition

$$\frac{\text{A polynomial}}{(s-a_1)^{n_1}...(s-a_m)^{n_m}(s^2+c_1s+d_1)^{k_1}...(s^2+c_ls+d_l)^{k_l}}$$

$$= \frac{A_1^1}{s-a_1} + \frac{A_2^1}{(s-a_1)^2} + ... + \frac{A_{n-1}^1}{(s-a_1)^{n_1}} +$$

$$\ldots + \frac{A_1^m}{s-a_m} + \frac{A_2^m}{(s-a_m)^2} + ... + \frac{A_{n-1}^m}{(s-a_m)^{n_m}}$$

$$\ldots + \frac{B_1^1 s + C_1^1}{s^2+c_ls+d_l} + \frac{B_2^1 s + C_2^1}{(s^2+c_ls+d_l^2)^2} + ... + \frac{B_{k_1}^1 + C_{k_1}^1}{(s^2+c_ls+d_l^1)^{k_1}}$$

$$+ \ldots + \frac{B_1^l s + C_1^l}{s^2+c_ls+d_l} + \frac{B_2^l s + C_2^l}{(s^2+c_ls+d_l^2)^2} + ... + \frac{B_{k_l}^l + C_{k_l}^l}{(s^2+c_ls+d_l^1)^{k_l}}$$

To find the coefficients in the numerator, take common denominator method. Then use a combination of plugging in each zero of the denominators and setting the coefficients of like terms equal to each other. You always set the numerator equal to the original numerator.

Find the Laplace inverse of the following functions:

Examples:

- $Y(s) = \dfrac{s^2 - 11s - 26}{s(s-1)(s+4)}$

  Solution: partial fraction decomposition gives:

  $$F(s) = \frac{\frac{13}{2}}{s} - \frac{\frac{36}{5}}{s-1} + \frac{\frac{17}{10}}{s+4}$$

  $$f(t) = \tfrac{13}{2} - \tfrac{36}{5}e^{t} + \tfrac{17}{10}e^{-4t}$$

- $\dfrac{5s^2 - 26s - 16}{(s-2)^3}$

  Solution:

  Partial functions gives:

  $$F(s) = \frac{5}{s-2} - \frac{6}{(s-2)^2} - \frac{48}{(s-2)^3}$$

  $$f(t) = 5e^{2t} - 6te^{2t} - 24t^2e^{2t}$$

- $\dfrac{s^2+10s+16}{(s-6)(s^2+16)}$

Solution:

the equations to solve:

$A+B=1$
$16A-6C=16$
$-6B+C=10$

$$\frac{\frac{28}{13}}{s-6}+\frac{-\frac{15}{13}s+\frac{40}{13}}{s^2+16}$$

$$\mathscr{L}^{-1}(F(s))=f(t)=\frac{28}{13}e^{6t}-\frac{15}{13}\cos(2t)+\frac{10}{13}\sin(2t)$$

- $\dfrac{(s^2+10s+16)e^{-2s}}{(s-6)(s^2+16)}$

Solution:

First find the inverse of

$\dfrac{(s^2+10s+16)}{(s-6)(s^2+16)}$ Using the previous problem:

$$\mathscr{L}^{-1}\left(\frac{(s^2+10s+16)}{(s-6)(s^2+16)}\right) = \frac{28}{13}e^{6t} - \frac{15}{13}\cos(2t) + \frac{10}{13}\sin(2t)$$

Then translate over x-axis.

$$\mathscr{L}^{-1}\left(\frac{(s^2+10s+16)e^{-2s}}{(s-6)(s^2+16)}\right) =$$

$$\left(\frac{28}{13}e^{6t-12} - \frac{15}{13}\cos(2t-4) + \frac{10}{13}\sin(2t-4)\right)u_2(t)$$

- $\dfrac{3s-1}{s^2+8s+20}$

  Solution: First use the denominator to make a complete square plus a constant.

  $s^2+8s+20 = (s+\frac{8}{2})^2 +$ something

  something is $s^2+8s+20-(s+\frac{8}{2})^2 = 4$

  Then rewrite:

  $$F(s) = \frac{3s-1}{(s+4)^2+4}$$

  Now we need to make the denominator in terms of $s+4$ and some constants:

So built $3(s+4)$ and subtract what is needed:

$$\frac{3s+12-11}{(s+4)^2+4}$$

separate the fractions:

$$\frac{3(s+4)}{(s+4)^2+4} - \frac{11}{(s+4)^2+4}$$

The solution is : $3e^{-4t}\cos(2t) - \frac{11}{2}e^{-4t}\sin(2t)$

Solving Linear ODE Using Laplace Transforms

- Take the Laplace transform of the equation. (Use the initial values to find $\mathscr{L}(y^{(n)}(t))$.)

- Solve for $Y(s)$ in the resulting linear equation.

- Use Laplace inverse methods to find $y(t)$.

Examples

- Solve $y'' - 5y' + 6y = 0 \qquad y(0) = 2,\ y'(0) = 7$

Let $Y(s) = \mathscr{L}(y(t))$.

The first step is to take the Laplace transform of both sides of the original differential equation. We have

$$\mathscr{L}\left[y'' - 5y' + 6y\right] = \mathscr{L}(0)$$

Obviously, $\mathscr{L}(0) = 0$ and

$$\mathscr{L}\left[y'' - 5y' + 6y\right](s) = \mathscr{L}[y''](s) - 5\mathscr{L}[y'](s) + 6\mathscr{L}[y](s)$$

Now use the formulas for the $\mathscr{L}(y'')$ and $\mathscr{L}(y')$:

$$\mathscr{L}(y') = s\mathscr{L}[y] - y(0) = sY(s) - 2.$$

$$\mathscr{L}[y''] = s^2\mathscr{L}[y] - sy(0) - y'(0) = s^2Y(s) - 2s - 7$$

Now $\mathscr{L}[y''](s) - 5\mathscr{L}[y'](s) + 6\mathscr{L}[y](s) = s^2Y(s) - 2s - 7 - 5sY(s) + 10 + 6Y(s)$

The Laplace-transformed differential equation is

$$(s^2 - 5s + 6)Y(s) - 2s - 3 = 0$$

Solve for $Y(s)$: $Y(s) = \dfrac{2s - 3}{s^2 - 5s + 6}$

Match the denominator to the table. Since the denominator can be factored, use partial fraction decomposition : $Y(s) = \frac{A}{s-2} + \frac{B}{s-3} = \frac{A(s-3)+B(s-2)}{(s-2)(s-3)} = \frac{2s-3}{(s-2)(s-3)}$

$s = 3$ gives $B = 3$.

$s = 2$ gives $A = -1$.

$$Y(s) = \frac{-1}{s-2} + \frac{3}{s-3}$$

Find the Laplace inverse in the table:

$$\mathscr{L}^{-1}\left[\frac{1}{s-2}\right](t) = e^{2t}$$

$$\mathscr{L}^{-1}\left[\frac{1}{s-3}\right](t) = e^{3t}$$

$$y(t) = \mathscr{L}\left[\frac{-1}{s-2} + \frac{3}{s-3}\right](t) = -e^{2t} + 3e^{3t}$$

This question was a homogeneous equation and easy to solve with other methods.

Example

- $y'' + 4y' + 5y = 10e^t \qquad y(0) = 1 \qquad y'(0) = 2$

$$\mathscr{L}(y'' + 4y' + 5y) = \mathscr{L}(10e^t)$$

The Laplace transform of the LHS $\mathscr{L}\left(y'' + 4y' + 5y\right)$ is

$$\mathscr{L}\left(y'' + 4y' + 5y\right) = \mathscr{L}(y'') + 4\mathscr{L}(y') + 5\mathscr{L}(y)$$

$$= s^2 Y(s) - sy(0) - y'(0) + 4sY(s) - 4y(0) + 5Y(s)$$

The Laplace transform of the RHS is

$$\mathscr{L}\left(10e^t\right) = \frac{10}{s-1}$$

Equating the LHS and RHS and using the fact that $y(0) = 1,\ y'(0) = 2$, we get

$$(s^2 + 4s + 5)Y(s) - s - 6 = \frac{10}{s-1}$$

Solving for $Y(s)$, we obtain:

$$Y(s) = \frac{s+6}{s^2+4s+5} + \frac{10}{(s-1)(s^2+4s+5)}$$

Use partial fraction decomposition :

$$\frac{10}{(s-1)(s^2+4s+5)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+4s+5}$$

$$= \frac{A(s^2+4s+5)+(Bs+C)(s-1)}{(s-1)(s^2+4s+5)}.$$

$s=1$ gives $A=1$.

$$As^2+4As+5A+Bs^2+(C-B)s-C=10$$

$$(A+B)s^2+(4A+C-B)s+(5A-C)=10$$

gives $C=-5$ and $B=-1$.

That is, $\dfrac{10}{(s-1)(s^2+4s+5)} = \dfrac{1}{s-1} + \dfrac{-s-5}{s^2+4s+5}$

So $Y(s) = \dfrac{1}{(s+2)^2+1} + \dfrac{1}{s-1}$

Use the table to find the inverse of $\dfrac{1}{(s-1)}$ is e^t and the inverse of $\dfrac{1}{(s+2)^2+1}$ is $e^{-2t}\sin(t)$, so

$y(t) = e^{-2t}\sin(t) + e^t$

More Examples: Solve the following equations.

- $$\begin{cases} y'' - 7y' + 6y = 4e^{2t} \\ y(0) = 6 \\ y'(0) = 1 \end{cases}$$

Solution:

$$s^2Y(s) - s(6) - 1 - 7(sY(s) - 6) + 6Y(s) = \frac{4}{s-2}$$

$$(s^2 - 7s + 6)Y(s) = \frac{4}{s-2} + 6s - 42 + 1$$

$$\boxed{Y(s) = \frac{4}{(s-2)(s^2-7s+6)} + \frac{6s-41}{s^2-7s+6}}$$

$$Y(s) = \frac{-1}{s-2} + (\tfrac{4}{5})\frac{1}{s-1} + (\tfrac{-4}{5})\frac{-1}{s-6} + \frac{7}{s-1}$$

$$Y(s) = \frac{-1}{s-2} + (\tfrac{39}{5})\frac{1}{s-1} + (-\tfrac{4}{5})(\frac{1}{s-6})$$

$$y(t) = \mathscr{L}^{-1}(Y(s)) = -e^{2t} + \frac{39}{5}e^{t} - \tfrac{4}{5}e^{6t}$$

- $$\begin{cases} y'' - 7y' + 6y = 4e^{t} \\ y(0) = 6 \\ y'(0) = 1 \end{cases}$$

Solution:

$$s^2Y(s) - s(6) - 1 - 7(sY(s) - 6) + 6Y(s) = \frac{4}{s-2}$$

$$(s^2 - 7s + 6)Y(s) = \frac{4}{s-2} + 6s - 42 + 1$$

$$\boxed{Y(s) = \frac{4}{(s-2)(s^2-7s+6)} + \frac{6s-41}{s^2-7s+6}}$$

$$Y(s) = (\tfrac{-1}{25})\frac{1}{s-1} + (\tfrac{-4}{5})\frac{1}{(s-1)^2} + (\tfrac{4}{5})\frac{-1}{s-6} + (\frac{7}{s-1} + \frac{-1}{s-6}$$

$$Y(s) = (\tfrac{171}{25})\frac{1}{s-1} + (\tfrac{-4}{5})\frac{1}{(s-1)^2} + (\tfrac{-21}{25})(\frac{1}{s-6})$$

$$y(t) = \mathscr{L}^{-1}(Y(s)) = -(\tfrac{171}{25})e^t + \frac{-4}{5}te^t - \tfrac{-21}{25}e^{6t}$$

- $$\begin{cases} y'' + 16y = \cos 4t \\ y(0) = 0 \\ y'(0) = 1 \end{cases}$$

Solution:

$$s^2Y(s) - 1 + 16Y(s) = \frac{s}{s^2+16}$$

$$(s^2+16)Y(s) = \frac{s}{s^2+16} + 1$$

$$\boxed{Y(s) = \frac{s}{(s^2+16)^2} + \frac{1}{s^2+16}}$$

$$Y(s) = (\frac{1}{8})\left(\frac{8s}{(s^2+16)^2}\right) + (\frac{1}{4})\left(\frac{4}{s^2+16}\right)$$

$$y(t) = \mathscr{L}^{-1}(Y(s)) = \frac{1}{8}t\sin 4t + \frac{1}{4}\sin 4t$$

Example

Solve

$$y'' + 4y' + 13y = 5e^t$$
$$y(0) = 1$$
$$y'(0) = 2$$

Solution

- Apply Laplace transform to the ode:

$$(s^2Y(s) - s - 2) + 4(sY(s) - 1) + 13Y(s) = \frac{5}{s-1}$$

- All terms with $Y(s)$ together:

$$(s^2 + 4s + 13)Y(s) - s - 6 = \frac{5}{s-1}$$

$$(s^2 + 4s + 13)Y(s) = s + 6 + \frac{5}{s-1}$$

So $Y(s) = \dfrac{s+6}{(s^2+4s+13)} + \dfrac{5}{(s-1)(s^2+4s+13)}$

- Notice: $(s^2 + 4s + 13)$ does not have real factors.

  The second fraction has two factors in the denominator so use the partial fraction decomposition :

$$\frac{5}{(s-1)(s^2+4s+13)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+4s+13}$$

  That is:

$$\frac{5}{(s-1)(s^2+4s+13)} = \frac{A(s^2+4s+13)+(Bs+C)(s-1)}{(s-1)(s^2+4s+13)}$$

  That is $\frac{5}{(s-1)(s^2+4s+13)} = \frac{(A+B)s^2+(4A-B+C)s+(13A}{(s-1)(s^2+4s+13)}$

- To find the coefficients, solve the system:

$$\begin{cases} A+B=0 \\ 4A-B+C=0 \\ 13A-C=5 \end{cases}$$

- That is, use your calculator to do:

$$rref\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 4 & -1 & 1 & 0 \\ 13 & 0 & -1 & 5 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{5}{18} \\ 0 & 1 & 0 & \frac{-5}{18} \\ 0 & 0 & 1 & \frac{-25}{18} \end{array}\right]$$

- So $A = \frac{5}{18}$, $B = \frac{-5}{18}$ and $C = \frac{-25}{18}$
- Rewrite $Y(s) = \frac{s+6}{(s^2+4s+13)} + \frac{\frac{5}{18}}{(s-1)} + \frac{\frac{-5}{18}s + \frac{-25}{18}}{(s^2+4s+13)}$
- Add the first and the last fraction to get:

  $$Y(s) = \frac{\frac{5}{18}}{(s-1)} + \frac{\frac{13}{18}s + \frac{83}{18}}{(s^2+4s+13)}$$
- Make the second denominator into a complete square: $s^2+4s+13 = (s+\frac{4}{2})^2+9$

  $$Y(s) = \frac{\frac{5}{18}}{(s-1)} + \frac{\frac{13}{18}s + \frac{83}{18}}{(s+2)^2+3^2}$$
- Use the table to make the numerator:

$$Y(s) = \frac{\frac{5}{18}}{(s-1)} + \frac{\frac{13}{18}(s+2) + \frac{83-13\times 2}{18\times 3}(3)}{(s+2)^2+3^2}$$

- So $Y(s) = \frac{\frac{5}{18}}{(s-1)} + \frac{\frac{13}{18}(s+2) + \frac{19}{18}(3)}{(s+2)^2+3^2}$

- Now:

$$\boxed{y(t) = \frac{5}{18}e^{t} + \frac{13}{18}e^{-2t}\cos(3t) + \frac{19}{18}e^{-2t}\sin(3t)}$$

A few details on calculating Laplace inverse of the part involving Heaviside function:

- Use the table to find the Laplace transform.

- Remember that to find Laplace transform of the term with Heaviside, $\left(f(t-c)u_c(t)\right)$, translate the function multiplying the u_c. $\left(f(t-c) \to f(t)\right)$ Then take the Laplace transform of the translated function: $F(s) = \mathscr{L}(f(t)$ and finally multiply that by e^{-cs} to get: $F(s)e^{-cs}$.

- Find $Y(s)$.

- In the term that contains e^{-cs}, ignore e^{-cs} and set the rest of the function to be $H(s)$

- Do the partial fraction decomposition on $H(s)$ if applicable.

- Find $h(t)$ using the table.

- Replace all ts in $h(t)$ by $(t-c)$ and name that $h(t-c)$. (Translate back.)

- The Laplace inverse of the term related to the Heaviside $u_c(t)$ is : $\boxed{h(t-c)u_c(t)}$

Heaviside Examples:

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$$\begin{cases} y'' + y = u_{3\pi}(t) \\ y(0) = 1 \\ y'(0) = 0 \end{cases}$$

Solution:

$$F(s) = \frac{e^{-3\pi s}}{s(s^2+1)} + \frac{s}{s^2+1}$$

$$F(s) = \frac{e^{-3\pi s}}{s} - \frac{se^{-3\pi s}}{s^2+1} + \frac{s}{s^2+1}$$

$$f(t) = u_{3\pi}(t)\Big(1 - \cos(t - 3\pi)\Big) + \cos(t)$$

- $$\begin{cases} y'' + y = u_0(t) - u_{3\pi}(t) \\ y(0) = 0 \\ y'(0) = 1 \end{cases}$$

Solution:

$$Y(s) = \frac{1}{s^2+1} + \frac{1}{s(s^2+1)} - \frac{e^{-3\pi s}}{s(s^2+1)}$$

$$Y(s) = \frac{1}{s^2+1} + \frac{1}{s} - \frac{s}{s^2+1} - \frac{e^{-3\pi s}}{s} + \frac{se^{-3\pi s}}{s^2+1}$$

$$\boxed{y(t) = \sin(t) + 1 - \cos(t) - u_{3\pi}(t) + \cos(t-3\pi)\,u_{3\pi}(t)}$$

or $y = 1 + \sin(t) - \cos(t) - (1+\cos(t))\,u_{3\pi}(t)$

- $$\begin{cases} y'' + y' + \frac{5}{4}y = u_0(t)\sin(t) + u_\pi(t)\sin(t-\pi) \\ y(0) = 0 \\ y'(0) = 0 \end{cases}$$

Solution:

$$Y(s) = \frac{1}{(s^2+1)(s^2+s+\frac{5}{4})} + \frac{e^{-\pi s}}{(s^2+1)(s^2+s+\frac{5}{4})}$$

$$Y(s) = (-\tfrac{16}{17})\frac{s}{s^2+1} + (\tfrac{4}{17})\frac{1}{s^2+1} + (\tfrac{16}{17})\frac{s}{(s+1/2)^2+1} + (\tfrac{12}{17})\frac{1}{s^2+1}$$

$$+\left((-\tfrac{16}{17})\frac{s}{s^2+1} + (\tfrac{4}{17})\frac{1}{s^2+1} + (\tfrac{16}{17})\frac{s+\frac{1}{2}}{(s+1/2)^2+1} + (\tfrac{4}{17})\frac{1}{(s+1/2)^2+1}\right)e^{-\pi s}$$

$$y = (-\tfrac{16}{17})\cos(t) + (\tfrac{4}{17})\sin(t) + (\tfrac{16}{17})\cos(t)e^{-t/2} + (\tfrac{4}{17})\sin(t)e^{-t/2} + u_\pi(t)\left(-\frac{16}{17}\cos(t-\pi) + \frac{4}{17}\sin(t-\pi) + \frac{16}{17}\cos(t-\pi)e^{-\frac{t}{2}+\frac{\pi}{2}} + \frac{4}{17}\sin(t-\pi)e^{-\frac{t}{2}+\frac{\pi}{2}}\right)$$

You will find many IVPs are easier to solve with Laplace transform. Resonance could be a bit tricky though:

- Solve

$$\begin{cases} y'' + y = \sin(t) \\ y(0) = 0 \\ y'(0) = 0 \end{cases}$$

Solution:

Note: This is resonance.

$$Y(s) = \frac{1}{(s^2+1)^2}$$

$$Y(s) = \frac{\frac{1}{2}}{s^2+1} - (\tfrac{1}{2})\frac{s^2-1}{(s^2+1)^2}$$

$$\boxed{\frac{1}{2}\sin(t) - \frac{1}{2}t\cos(t)}$$

Laplace Table:

1	$\frac{1}{s}$	$s > 0$
t	$\frac{1}{s^2}$	$s > 0$
t^n	$\frac{n!}{s^{n+1}}$	$s > 0, n = integer$
e^{at}	$\frac{1}{s-a}$	$s > 0, a = constant$
$\cos bt$	$\frac{s}{(s^2+b^2)}$	$s > 0, b = constant$
$\sin bt$	$\frac{b}{s^2+b^2}$	$s > 0, b = constant$
$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$	$s > a$
$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$	$s > a$

$t\cos bt$	$\dfrac{s^2-b^2}{(s^2+b^2)^2}$	$s>0$
$t\sin bt$	$\dfrac{2bs}{(s^2+b^2)^2}$	$s>0$
$t^n f(t)$	$(-1)^n\dfrac{d^nF(s)}{ds^n}$	
$u_c(t)$	$\dfrac{e^{-cs}}{s}$	
$u_c(t)h(t-c)$	$e^{-cs}H(s)$	
y'	$sY(s)-y(0)$	
y''	$s^2Y(s)-sy(0)-y'(0)$	
$f(t)e^{at}$	$F(s-a)$	$s>a$
te^{at}	$\dfrac{1}{(s-a)^2}$	$s>a$
t^2e^{at}	$\dfrac{2!}{(s-a)^3}$	$s>a$