Eigenvalues/ Eigenvectors.

Eigenvalues and eigenspaces are important elements of many fields. One of the early questions that contributed to their fame was solving electrical networks which results in solving linear systems of differential equations.

The definitions

Definition

A number[a] λ is called an **eigenvalue** of an $n \times n$ matrix A, if there exist a non-zero column vector $\vec{\xi}$ such that $A\vec{\xi} = \lambda\vec{\xi}$. Then the column vector is called **eigenvector**.

[a] First we discuss real numbers. Then we discuss complex numbers.

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Example: Consider the matrix

$A = \begin{bmatrix} 2 & 2 & 3 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$. Note that $\begin{bmatrix} 2 & 2 & 3 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 0 \end{bmatrix} = 3 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ So $\lambda = 3$ is an eigenvalue corresponding to the eigenvector $\vec{\xi} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$.

[1] λ reads lambda. ξ reads xi.

The Constructive Definition

The definition in the previous page does not explain how to find the eigenvalues of a matrix. The following gives a method of finding the eigenvalue.

Definition

For $n \times n$ matrix A, $\det(A - xI_{n\times n})$ is a polynomial of degree n and is called the **characteristic polynomial** of A.
The zeros of the characteristic polynomial $\det(A - xI_{n\times n}) = 0$ are the **eigenvalues** of A.

$\Longrightarrow$: Let λ be a zero of the polynomial $\det(A - xI_{n\times n}) = 0$, then the matrix $A - \lambda I_{n\times n}$ is a singular matrix. That is, $(A - \lambda I_{n\times n})\vec{\mathbf{x}} = \vec{\mathbf{0}}$ has infinite solutions. Let $\vec{\xi}$ be one of those solutions. Then $(A - \lambda I_{n\times n})\vec{\xi} = \vec{\mathbf{0}} \Longrightarrow A\vec{\xi} - \lambda\vec{\xi} = \vec{\mathbf{0}} \Longrightarrow A\vec{\xi} = \lambda\vec{\xi}$ which shows that $\vec{\xi}$ is an eigenvector corresponding to the eigenvalue λ.
$\Longleftarrow$: If λ and $\vec{\xi}$ are such that $A\vec{\xi} = \lambda\vec{\xi}$. Then $(A - \lambda I_{n\times n})\vec{\xi} = \vec{\mathbf{0}}$ so $(A - \lambda I_{n\times n})$ is singular and $\det((A - \lambda I_{n\times n})) = 0$.

Definition

Definition

Let λ be an eigenvalue of A, the null space of $A - \lambda I$ is called the **eigenspace** of A corresponding to λ.
That is, the set of all eigenvectors corresponding to λ union with $\{\vec{\mathbf{0}}\}$.

Example

Example

Find all eigenvalues of A and the eigenvectors corresponding to them.

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

Set $\det(A - \lambda I) = 0$ That is, $\det\left(\begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 4 \\ 0 & 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0 \implies$

$$\det\left(\begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 4 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}\right) = 0 \implies \det\left(\begin{bmatrix} 2-\lambda & 0 & 1 \\ 0 & 3-\lambda & 4 \\ 0 & 0 & 1-\lambda \end{bmatrix}\right) = 0$$

$$\implies (2-\lambda)(3-\lambda)(1-\lambda) = 0 \implies \lambda = 1, 2, 3$$

Example Continued

- For $\lambda = 1$, $(A - (1)I_{3\times 3})\vec{\xi} = \vec{\mathbf{0}}$ gives

$$\begin{bmatrix} 2-1 & 0 & 1 \\ 0 & 3-1 & 4 \\ 0 & 0 & 1-1 \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\xrightarrow{rref} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Longrightarrow \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \begin{bmatrix} -t \\ -2t \\ t \end{bmatrix}$$

That is, an eigenvector corresponding to $\lambda = 1$ is a non-zero multiple of $\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$

Example Continued

- For $\lambda = 2$, $(A - (2)I_{3\times3})\vec{\xi} = \vec{\mathbf{0}}$ gives

$$\begin{bmatrix} 2-2 & 0 & 1 \\ 0 & 3-2 & 4 \\ 0 & 0 & 1-2 \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\xrightarrow{rref} \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Longrightarrow \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- For $\lambda = 3$, $(A - (3)I_{3\times3})\vec{\xi} = \vec{\mathbf{0}}$ gives

$$\begin{bmatrix} 2-3 & 0 & 1 \\ 0 & 3-3 & 4 \\ 0 & 0 & 1-3 \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 4 \\ 0 & 0 & -2 \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\xrightarrow{rref} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Longrightarrow \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Example Continued

So the eigenspaces are:

$$E_1 = \left\{ k \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} \middle| k \in \mathbb{R} \right\}$$

$$E_2 = \left\{ k \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \middle| k \in \mathbb{R} \right\}$$

$$E_3 = \left\{ k \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \middle| k \in \mathbb{R} \right\}$$

Note that each subspace is one dimensional.

Theorem

The previous example helps understanding this theorem.

Theorem

let A be an $n \times n$ upper or lower triangular matrix, then the eigenvalues of A are exactly the entries on the main diagonal of A.

Eigenvalues Using the Calculator, TI84

Learning the process of finding characteristic polynomial is important and is emphasized on the homework assignment. But we also would like to see an easier way of finding eigenvalues using computational tool that we all have available. Here is the process:

1. Enter the $n \times n$ matrix whose eigenvalue you are evaluating in A. [2]
2. Press Y= and enter the following in $\backslash Y_1 =$, using the MATRIX menu. $\det([A] - Xidentity(n))$ where n is the size of the square matrix.
3. Graph and find all zeros of the function.

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[2] This is done by pressing 2nd and MATH which wil give access to MATRIX . Then choose EDIT and choose matrix A. Enter the dimensions of the matrix. Then enter all entries.

[3] Please note that $[A]$ is the key for matrix A is presented under matrix menu.

How to Recognize the multiplicity of zeros.

1. If x-axis is tangent to the graph at the zero but the graph is not passing through x-axis at the zero then zero is of **even** multiplicity.
2. If the x-axis is tangent to the graph and the graph is passing through x-axis, the zero is of **odd** multiplicity > 1.
3. Otherwise, if the x-axis is not tangent to the graph, the zero is simple.
4. To find the complex roots for **quadratic** equations only, x-value of the non-zero min or max will give the real part and y-value of the non-zero min or max gives the imaginary part.[4]

[4] This only is useful for 2×2 matrices.

EigenValues of Higher Multiplicity.

Example

Find eigenspaces of the following matrix

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$\det(A - \lambda I) = 0$ gives $A = \det \begin{bmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{bmatrix} = -\lambda \det \begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} -$
$(1) \det \begin{pmatrix} 1 & 1 \\ 1 & -\lambda \end{pmatrix} + (1) \det \begin{pmatrix} 1 & -\lambda \\ 1 & 1 \end{pmatrix} = -\lambda^3 + \lambda + \lambda + 1 + 1 + \lambda = -\lambda^3 + 3\lambda + 2$

The characteristic polynomial is $-\lambda^3 + 3\lambda + 2 = -(\lambda + 1)^2(\lambda - 2)$.
factor the polynomial or use your calculator to find the roots. The eigenvalues are -1 (repeated) and 2.

Continued Example

- For $\lambda = -1$, solve $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

  The rref of the augmented matrix: $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ This gives the following solution set which is the eigenspace corresponding to $\lambda = -1$.

  $E_{-1} = \left\{ \begin{bmatrix} -s-t \\ s \\ t \end{bmatrix} \middle| s \in \mathbb{R},\ t \in \mathbb{R} \right\}$. Any non-zero vector in this vector space is an eigenvector corresponding to $\lambda = -1$.

  We are usually interested in fining a basis for the eigenspace.

  $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ which means that the eigenspace is two dimensional. [5]

[5] $\lambda = -1$ was a root of multiplicity 2 in the characteristic equation and corresponding eigenspace was of higher dimension too. Note that this is not always the case.

Continued Example

- For $\lambda = 2$ solve $\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

  The rref of the augmented matrix: $\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$.

  the eigenspace corresponding to $\lambda = 2$.

  $E_2 = \left\{ \begin{bmatrix} t \\ t \\ t \end{bmatrix} \middle| \; t \in \mathbb{R} \right\}$. Any non-zero vector in this vector space is an eigenvector corresponding to $\lambda = -1$.

  We are usually interested in fining a basis for the eigenspace.

  $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ which means that the eigenspace is one dimensional.

General form for a 3×3 matrix:

For matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, let λ be such that

$\det(A - \lambda I_{3\times3}) = 0$ That is, $\begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda \end{vmatrix} = 0.$

What are the **eigenvectors** corresponding to an eigenvalue?

If λ is an eigenvalue of matrix A then, any non-trivial solution to $\left(A - \lambda \mathbf{I_{3\times3}}\right)\vec{\xi} = \vec{0}$ is an eigenvector corresponding to λ.

That is, a non-trivial solution to $\begin{bmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

is an eigenvector corresponding to λ and the solution set is the eigenspace corresponding λ.

A Few Notes

- A is singular if and only if $\lambda = 0$ is an eigenvalue of A.
- $\vec{\mathbf{0}}$ is not an eigenvector for any matrix. This is understandable because we are not looking for trivial solution.

Definitions and Facts on Multiple eigenvalues.

Definition

Let $\{\lambda_1, \lambda_2, \cdots, \lambda_k\}$ be the complete list of eigenvalues for A (including all repeated eigenvalues) then,

1. If λ occurs only once in the list then we call λ **simple**.
2. If λ occurs k times in the list then we say that λ has **algebraic multiplicity** k.
3. If k is the dimension of the eigenspace corresponding to λ, then the **geometric multiplicity** of λ is k.
4. The geometric multiplicity $\leq$ algebraic multiplicity for any eigenvalue λ. That is, if λ is simple, the dimension of the corresponding eigenspace is 1. If λ is an eigenvalue of algebraic multiplicity $k > 1$, then the dimension of the corresponding eigenspace is anywhere between and including one and k.

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[6] These facts and what follow are also true if the eigenvalue is a complex number.

Theorem

Theorem

For square matrix A, let $\vec{\xi}^1$ *,* $\vec{\xi}^2$*,...,* $\vec{\xi}^k$ *be the eigenvectors corresponding to distinct eigenvalues* λ_1*,...* λ_k*, respectively, then the eigenvectors are all* **linearly independent**.

The Geometric Multiplicity

Theorem

Let λ be an eigenvalue of matrix A of algebraic multiplicity $k > 1$, then the null space of $(A - \lambda I)^k$ is k dimensional.

In different fields, there is a need to find a spacial basis for $N[(A - \lambda I)^k]$. This is a basis that includes a basis for $N[A - \lambda I]$.

Definition

Non-zero elements of null space of $(A - \lambda I)^k$ that are not in the null space of $(A - \lambda I)$ are called **generalized eigenvectors.**

Note that a generalized eigenvector $\vec{\eta}$ satisfies $(A - \lambda I)^k \vec{\eta} = \vec{\mathbf{0}}$ for $k > 1$ but $(A - \lambda I)\vec{\eta} \neq \vec{\mathbf{0}}$. (In some text the last condition may be removed to allow an eigenvector be also a generalized eigenvector.)

The Process of Finding Generalized Eigenvectors

Let λ be an eigenvalue of multiplicity $k > 1$. To satisfy specific properties, for every eigenvector $\vec{\xi}$ corresponding to λ, we are looking for generalized eigenvectors $\{\vec{\eta}_1, \vec{\eta}_2, ...\vec{\eta}_m\}$[7], if they exist, such that $(A - \lambda I)\vec{\eta}_1 = \vec{\xi}$ or $(A - \lambda I)\vec{\eta}_j = \vec{\eta}_i$. [8]
Please note that the dimension of some of the null spaces of $(A - \lambda I)$ is more that k.

[7] η reads "eta".

[8] Note that if $(A - \lambda I)^k \vec{\eta}_i = \vec{\mathbf{0}}$.

Example of Finding Generalized Eigenvectors

Example

Find the generalized eigenvectors of the matrix $A = \begin{bmatrix} 1 & 3 & 6 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix}$ for $\lambda = 1$.

Solve for the eigenspace corresponding to $\lambda = 1$:

$$\text{rref}\left(\left[\begin{array}{ccc|c} 0 & 3 & 6 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]\right) = \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \implies \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} \implies \left\{\begin{bmatrix} \color{red}{1} \\ \color{red}{0} \\ \color{red}{0} \end{bmatrix}\right\} \text{ Solve for}$$

$$\begin{bmatrix} 0 & 3 & 6 \\ 0 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \color{red}{1} \\ \color{red}{0} \\ \color{red}{0} \end{bmatrix} \implies \text{rref}\left(\left[\begin{array}{ccc|c} 0 & 3 & 6 & 1 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]\right) = \left[\begin{array}{ccc|c} 0 & 1 & 0 & 1/3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{bmatrix} t \\ 1/3 \\ 0 \end{bmatrix} \xrightarrow{\text{Choose a solution.}} \begin{bmatrix} 0 \\ 1/3 \\ 0 \end{bmatrix} \text{ is a generalized eigenvector. } \left\{\begin{bmatrix} 0 \\ 1/3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}$$

is a basis for the null space of $(A - \lambda I)^2$.

Example of Finding Generalized Eigenvectors

Example

Find all linearly independent generalized eigenvectors for matrix
$A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 2 \\ 0 & -1 & 1 \end{bmatrix}$

First solve for $\det(A - \lambda I) = 0$ and find the eigenvalues of A.
The only eigenvalue of this matrix is $\lambda = 1$. Find a basis for its eigenspace.
That is, find a basis for $N(A - I)$: $\text{rref}\left(\left[\begin{array}{ccc|c} 0 & 2 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right]\right) = \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

The solution is $\begin{bmatrix} -2t \\ 0 \\ t \end{bmatrix}$. That is, $\left\{\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}\right\}$ is a basis for the eigenspace.

Continued Example

Solve for

$$\begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 2 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \implies \text{rref}\left(\left[\begin{array}{ccc|c} 0 & 2 & 0 & -2 \\ 1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 1 \end{array}\right]\right) = \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The general solution is $\begin{bmatrix} -2t \\ -1 \\ t \end{bmatrix}$, choose as many LI vectors as you can.

I choose $\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$. Then solve for $\begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 2 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \implies$

$$\text{rref}\left(\left[\begin{array}{ccc|c} 0 & 2 & 0 & 0 \\ 1 & 0 & 2 & -1 \\ 0 & -1 & 0 & 0 \end{array}\right]\right) = \left[\begin{array}{ccc|c} 1 & 0 & 2 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \implies \begin{bmatrix} -1-2t \\ 0 \\ t \end{bmatrix} \implies \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

Continues Example

$$B = \left\{ \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$$ is a basis for $(A - \lambda I)^3$.

Note that $(A - \lambda I)^3 = 0_{3\times 3}$[9] and B is a spacial basis for $\mathbb{R}^3$.

[9] Verify using your calculator.