The non-homogeneous linear equations are not going to be on the exam

Non-homogeneous linear systems of ode:

The systems of the form $\boxed{\vec{x}' = A\vec{x} + \vec{g}(t)}$ or

$$\boxed{\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} g_1(t) \\ g_2(t) \end{bmatrix}}$$

First step is to find the solutions to the homogeneous system $\vec{x}' = A\vec{x}$. Then use one of the following methods to find the particular solution $\vec{x}_p$. (Again The variation of parameters works for any forcing function. But simple exponentials, sine, cosine and polynomial forcing functions only can be solved using undetermined coefficients.)

- **Variation of parameters:**

  Let $\psi(t)$ be a matrix whose columns are the solution to the homogeneous system $\vec{x}' = A\vec{x}$.

  That is for distinct real roots λ_1 and λ_1:

  $$\psi(t) = \begin{bmatrix} \xi_1^1 e^{\lambda_1 t} & \xi_1^2 e^{\lambda_2 t} \\ \\ \xi_2^1 e^{\lambda_1 t} & \xi_2^2 e^{\lambda_2 t} \end{bmatrix}$$

  For repeated root λ:

$$\psi(t) = \begin{bmatrix} \xi_1 e^{\lambda t} & \xi_1 t e^{\lambda t} + \eta_1 e^{\lambda t} \\ \\ \xi_2 e^{\lambda t} & \xi_2 e^{\lambda t} + \eta_2 e^{\lambda t} \end{bmatrix}$$

And for complex roots $\lambda = \mu \pm \omega i$:

$$\psi(t) = \begin{bmatrix} e^{\mu t}\Big(\Re(\xi_1)\cos(\omega t) - \Im(\xi_1)\sin(\omega t)\Big) & e^{\mu t}\Big(\Re(\xi_1)\sin(\omega t) + \Im(\xi_1)\cos(\omega \\ \\ e^{\mu t}\Big(\Re(\xi_2)\cos(\omega t) - \Im(\xi_2)\sin(\omega t)\Big) & e^{\mu t}\Big(\Re(\xi_2)\sin(\omega t) + \Im(\xi_2)\cos(\omega \end{bmatrix}$$

Then since the columns are the solution to the homogeneous solution, $\psi(t)' = A\psi(t)$. $*$

The method of variation of parameter relies on the fact that The particular solution to the nonhomogeneous system is $\vec{x_p} = \vec{x_h}\vec{u(t)}$ where $\vec{u}$ is a vector.

To find $\vec{u}$, remember that $\psi(t)\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$ can be plug in the nonhomogeneous solution and satisfy the equation:

$$\left(\psi(t)\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}\right)' = A\left(\psi(t)\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}\right) + \begin{bmatrix} g_1(t) \\ g_2(t) \end{bmatrix}$$

Use product rule on the left hand side to rewrite the equation:

$$\Big(\psi(t)\Big)'\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} + \psi(t)\left(\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}\right)' = \Big(A\psi(t)\Big)\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} + \begin{bmatrix} g_1(t) \\ g_2(t) \end{bmatrix}$$

Using $*$, rewrite the equation again:

$$\psi(t)\left(\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}\right)' = \begin{bmatrix} g_1(t) \\ g_2(t) \end{bmatrix} \quad \text{or} \quad \boxed{\psi(t)\left(\begin{bmatrix} u_1'(t) \\ u_2'(t) \end{bmatrix}\right) = \begin{bmatrix} g_1(t) \\ g_2(t) \end{bmatrix}} \; (**)$$

So to solve for u_1 and u_2, first solve for u_1' and u_2' in equation

$(**)$ and then integrate.

Then $\boxed{\vec{x}_p = \psi(t)\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}}$

The solution to the non-homogeneous system is

$$\boxed{\vec{x} = \vec{x}_h + \vec{x}_p = C_1\vec{x}_{h1} + C_2\vec{x}_{h2} + \vec{x}_p}$$

- **Method of undetermined coefficients:**

  In this method depending on the functions that appear in the homogeneous solution make a modified guess "x_{p1}" for g_1 and "x_{p2}" for g_2. Plug in the modified guesses into the nonhomogenous system to find the coefficients. Plug in the coefficients into the modified guesses to find the $\vec{x}_p$.

  The solution to the non-homogeneous system is

  $$\boxed{\vec{x} = \vec{x}_h + \vec{x}_p = C_1\vec{x}_{h1} + C_2\vec{x}_{h2} + \vec{x}_p}$$

Example:

$$\vec{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}\vec{x} + \begin{bmatrix} 2e^t \\ -e^t \end{bmatrix}$$

Solution

Using Variation of parameters:

- First find the solutions to the homogeneous equation $\vec{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}\vec{x}$

- $\begin{vmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{vmatrix} = 0$ to get $\lambda^2 - 2\lambda - 3 = 0$ to get $\lambda_1 = -1$ and $\lambda_2 = 3$.

- For $\lambda = -1$, solve $\begin{bmatrix} 1-(-1) & 1 \\ 4 & 1-(-1) \end{bmatrix} \begin{bmatrix} \xi_1^1 \\ \xi_2^1 \end{bmatrix} = \vec{0}$ to get $\begin{bmatrix} -\frac{1}{2}k \\ k \end{bmatrix}$

  and choose one eigen vector $\boxed{\begin{bmatrix} -1 \\ 2 \end{bmatrix}}$ (By setting $k = 2$).

  So $\boxed{\vec{x}_h^1(t) = \begin{bmatrix} -e^{-t} \\ 2e^{-t} \end{bmatrix}}$ is the first solution.

- For $\lambda = 3$, solve $\begin{bmatrix} 1-(3) & 1 \\ 4 & 1-(3) \end{bmatrix} \begin{bmatrix} \xi_1^2 \\ \xi_2^2 \end{bmatrix} = \vec{0}$ to get $\begin{bmatrix} \frac{1}{2}k \\ k \end{bmatrix}$ and

  choose one eigen vector $\boxed{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}$ (By setting $k = 2$).

  So $\boxed{\vec{x}_h^2(t) = \begin{bmatrix} e^{3t} \\ 2e^{3t} \end{bmatrix}}$ is the second solution.

- So $\psi(t) = \begin{bmatrix} -e^{-t} & e^{3t} \\ 2e^{-t} & 2e^{3t} \end{bmatrix}$

- Use the $(**)$ formula to set up: $\begin{bmatrix} -e^{-t} & e^{3t} \\ 2e^{-t} & 2e^{3t} \end{bmatrix} \begin{bmatrix} u_1'(t) \\ u_2'(t) \end{bmatrix} = \begin{bmatrix} 2e^{t} \\ -e^{t} \end{bmatrix}$

- In case of higher dimensions, I suggest using reduced form echelon form to solve for u_1' and u_2'. But for 2×2, I suggest using the inverse of $\psi(t)$ to solve for u_1' and u_2'.

- Remember $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ab - cd} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ so

$$\psi(t)^{-1} = \frac{1}{(2e^{3t})(-e^{-t}) - (2e^{-t})(e^{3t})} \begin{bmatrix} 2e^{3t} & -e^{3t} \\ -2e^{-t} & -e^{-t} \end{bmatrix}$$

$$= -\frac{1}{4}e^{-2t} \begin{bmatrix} 2e^{3t} & -e^{3t} \\ -2e^{-t} & -e^{-t} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}e^{t} & \frac{1}{4}e^{t} \\ \frac{1}{2}e^{-3t} & \frac{1}{4}e^{-3t} \end{bmatrix}$$

- Now $\begin{bmatrix} u_1'(t) \\ u_2'(t) \end{bmatrix} = \psi(t)^{-1} \begin{bmatrix} 2e^{t} \\ -e^{t} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}e^{t} & \frac{1}{4}e^{t} \\ \frac{1}{2}e^{-3t} & \frac{1}{4}e^{-3t} \end{bmatrix} \begin{bmatrix} 2e^{t} \\ -e^{t} \end{bmatrix} = \begin{bmatrix} -\frac{5}{4}e^{2t} \\ \frac{3}{4}e^{-2t} \end{bmatrix}$

- Integrate u_1' and u_2' to get u_1 and u_2:

$$\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} \int -\frac{5}{4}e^{2t}\, dt \\ \int \frac{3}{4}e^{-2t}\, dt \end{bmatrix} = \begin{bmatrix} -\frac{5}{8}e^{2t} \\ -\frac{3}{8}e^{-2t} \end{bmatrix}$$

- Find the particular solution

$$\vec{x}_p(t) = \psi(t) \begin{bmatrix} -\frac{5}{8}e^{2t} \\ -\frac{3}{8}e^{-2t} \end{bmatrix} = \begin{bmatrix} -e^{-t} & e^{3t} \\ 2e^{-t} & 2e^{3t} \end{bmatrix} \begin{bmatrix} -\frac{5}{8}e^{2t} \\ -\frac{3}{8}e^{-2t} \end{bmatrix} = \begin{bmatrix} \frac{1}{4}e^{t} \\ -2e^{t} \end{bmatrix}$$

- The solution is $\vec{x}(t) = C_1 \begin{bmatrix} -e^{-t} \\ 2e^{-t} \end{bmatrix} + C_2 \begin{bmatrix} e^{3t} \\ 2e^{3t} \end{bmatrix} + \begin{bmatrix} \frac{1}{4}e^{t} \\ -2e^{t} \end{bmatrix}$

Now lets do the same example using the method of undetermined coefficients. $\vec{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \vec{x} + \begin{bmatrix} 2e^{t} \\ -e^{t} \end{bmatrix}$

Solution

- Repeat the process in example above to find the the homogeneous solution:

  $\vec{x}_h(t) = C_1 \begin{bmatrix} -e^{-t} \\ 2e^{-t} \end{bmatrix} + C_2 \begin{bmatrix} e^{3t} \\ 2e^{3t} \end{bmatrix}$

- Now the guess $\vec{y}_p$ for $\begin{bmatrix} 2e^{t} \\ -e^{t} \end{bmatrix}$ is $\vec{x}_p = \begin{bmatrix} Ae^{t} \\ Be^{t} \end{bmatrix}$ since all entries of $\vec{g(t)}$ are “different" functions from the entries of $\vec{x}_h(t)$.

- Take the derivative of the guess. In this case: $\vec{x'}_p = \begin{bmatrix} Ae^{t} \\ Be^{t} \end{bmatrix}$

- Plug in the system.

  $\begin{bmatrix} Ae^{t} \\ Be^{t} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} Ae^{t} \\ Be^{t} \end{bmatrix} + \begin{bmatrix} 2e^{t} \\ -e^{t} \end{bmatrix}$

  That is

  $\begin{bmatrix} Ae^{t} \\ Be^{t} \end{bmatrix} = \begin{bmatrix} Ae^{t} + Be^{t} \\ 4Ae^{t} + Be^{t} \end{bmatrix} + \begin{bmatrix} 2e^{t} \\ -e^{t} \end{bmatrix}$

That is, $\begin{bmatrix} Ae^t \\ Be^t \end{bmatrix} = \begin{bmatrix} Ae^t + Be^t + 2e^t \\ 4Ae^t + Be^t + e^t \end{bmatrix}$

Subtract the left-hand side from both side of the equation:
$\begin{bmatrix} Be^t + 2e^t \\ 4Ae^t - e^t \end{bmatrix} = \vec{0}$

- That is $\begin{bmatrix} (B+2)e^t \\ (4A-1)e^t \end{bmatrix} = \vec{0}$
- $B+2=0$ and $4A-1=0$ $B=-2$ and $A=\dfrac{1}{4}$
- The solution is $\vec{x}(t) = C_1 \begin{bmatrix} -e^{-t} \\ 2e^{-t} \end{bmatrix} + C_2 \begin{bmatrix} e^{3t} \\ 2e^{3t} \end{bmatrix} + \begin{bmatrix} \frac{1}{4}e^t \\ -2e^t \end{bmatrix}$